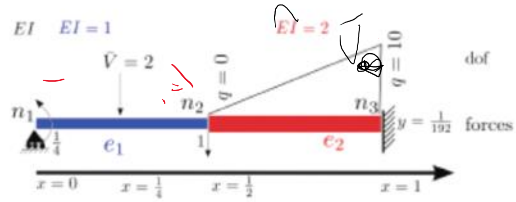
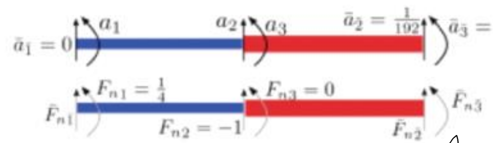
FEM20241113

Wednesday, November 13, 2024 9:26 PM

Beam example: Assembly of global system





1D elements

$$f^e = f_r^e + f_N^e - f_D^e$$

$$f_D^e = k^e a^e$$

e	e_1	e_2
$\mathbf{k}^e$	$\mathbf{k}^{e_1} = \frac{1}{(\frac{1}{2})^3}\begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \\ 3 & 0.5 & -3 & 1 \end{bmatrix}$ $= \begin{array}{c\|c} & \bar{1}\ \ 1\ \ 2\ \ 3 \\ \hline \begin{matrix}\bar{1}\\1\\2\\3\end{matrix} & \begin{bmatrix} 96 & 24 & -96 & 24 \\ 24 & 8 & -24 & 4 \\ -96 & -24 & 96 & -24 \\ 24 & 4 & -24 & 8 \end{bmatrix}\end{array}$	$\mathbf{k}^{e_2} = \frac{2}{(\frac{1}{2})^3}\begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \\ 3 & 0.5 & -3 & 1 \end{bmatrix}$ $= \begin{array}{c\|c} & 2\ \ 3\ \ \bar{2}\ \ \bar{3} \\ \hline \begin{matrix}2\\3\\\bar{2}\\\bar{3}\end{matrix} & \begin{bmatrix} 192 & 48 & -192 & 48 \\ 48 & 16 & -48 & 8 \\ -192 & -48 & 192 & -48 \\ 48 & 8 & -48 & 16 \end{bmatrix}\end{array}$
$\mathbf{f}_r^e$	(440) (1st eqn) ($\xi_0 = 0$) $\bar{V}\begin{bmatrix} N_1^{e_1}(\xi_0) \\ N_2^{e_1}(\xi_0) \\ N_3^{e_1}(\xi_0) \\ N_4^{e_1}(\xi_0) \end{bmatrix} = -2\begin{bmatrix} \frac{1}{2} \\ \frac{\frac{1}{2}}{8} \\ \frac{1}{2} \\ -\frac{\frac{1}{2}}{8} \end{bmatrix} = \begin{matrix}\bar{1}\\1\\2\\3\end{matrix}\begin{bmatrix} -1 \\ -\frac{1}{8} \\ -1 \\ \frac{1}{8} \end{bmatrix}$	equation (433); $\mathbf{r}^e\,[q_1 \quad q_2]^T$ $\frac{1}{2}\begin{bmatrix} \frac{7}{20} & \frac{3}{20} \\ \frac{1}{40} & \frac{1}{60} \\ \frac{3}{20} & \frac{7}{20} \\ -\frac{1}{60} & -\frac{1}{40} \end{bmatrix}\begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{matrix}2\\3\\\bar{2}\\\bar{3}\end{matrix}\begin{bmatrix} \frac{3}{4} \\ \frac{1}{12} \\ \frac{7}{4} \\ -\frac{1}{8} \end{bmatrix}$
$\mathbf{f}_D^e$	$\mathbf{k}^{e_1}\mathbf{a}_1^e =$ $\begin{bmatrix} 96 & 24 & -96 & 24 \\ 24 & 8 & -24 & 4 \\ -96 & -24 & 96 & -24 \\ 24 & 4 & -24 & 8 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{matrix}1\\\bar{1}\\2\\3\end{matrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\mathbf{k}^{e_2}\mathbf{a}_2^e =$ $\begin{bmatrix} 192 & 48 & -192 & 48 \\ 48 & 16 & -48 & 8 \\ -192 & -48 & 192 & -48 \\ 48 & 8 & -48 & 16 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \frac{1}{192} \\ 0 \end{bmatrix} = \begin{matrix}2\\3\\\bar{2}\\\bar{3}\end{matrix}\begin{bmatrix} -1 \\ -\frac{1}{4} \\ 1 \\ -\frac{1}{4} \end{bmatrix}$
$\mathbf{f}_e^e$	$\mathbf{f}_e^{e_1} = \mathbf{f}_r^{e_1} + \cancel{\mathbf{f}_N^{e_1}} - \mathbf{f}_D^{e_1} = \begin{matrix}\bar{1}\\1\\2\\3\end{matrix}\begin{bmatrix} -1 \\ -\frac{1}{8} \\ -1 \\ \frac{1}{8} \end{bmatrix}$	$\mathbf{f}_e^{e_2} = \mathbf{f}_r^{e_2} + \cancel{\mathbf{f}_N^{e_2}} - \mathbf{f}_D^{e_2} = \begin{matrix}2\\3\\\bar{2}\\\bar{3}\end{matrix}\begin{bmatrix} \frac{7}{4} \\ \frac{1}{3} \\ \frac{3}{4} \\ \frac{1}{8} \end{bmatrix}$

$$\mathbf{K} = \begin{bmatrix} 8 & -24 & 4 \\ & 96+192 & -24+48 \\ \text{sym.} & & 8+16 \end{bmatrix} = \begin{bmatrix} 8 & -24 & 4 \\ & 288 & 24 \\ \text{sym.} & & 24 \end{bmatrix}$$

$$\mathbf{F} = \mathbf{F}_n + \mathbf{F}_e = \begin{bmatrix} \frac{1}{4} \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8} \\ -1+\frac{7}{4} \\ \frac{1}{8}+\frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} \\ -\frac{3}{4} \\ \frac{11}{24} \end{bmatrix} \Rightarrow$$

$$\mathbf{k}^e = \frac{EI}{L^{e3}}\begin{bmatrix} 12 & 6L^e & -12 & 6L^e \\ & 4L^{e2} & -6L^e & 2L^{e2} \\ & & 12 & -6L^e \\ \text{sym.} & & & 4L^{e2} \end{bmatrix} \quad \text{for constant } E \text{ and } I \qquad (427)$$

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$$\mathbf{f}_r^e \approx \mathbf{r}^e\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad \text{where } \mathbf{r}^e = L^e\begin{bmatrix} \frac{7}{20} & \frac{3}{20} \\ \frac{1}{20}L^e & \frac{1}{30}L^e \\ \frac{3}{20} & \frac{7}{20} \\ -\frac{1}{30}L^e & -\frac{1}{20}L^e \end{bmatrix} \quad \text{exact for linear } q \qquad (433)$$

$$\mathbf{K} = \begin{bmatrix} 8 & -24 & 4 \\ & 96+192 & -24+48 \\ \text{sym.} & & 8+16 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -24 & 4 \\ & 288 & 24 \\ \text{sym.} & & 24 \end{bmatrix}$$

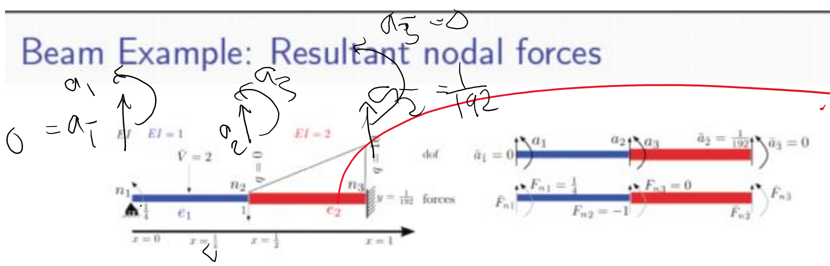
$$\mathbf{F} = \mathbf{F}n + \mathbf{F}e$$

$$= \begin{bmatrix} \frac{1}{4} \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8} \\ -1+\frac{7}{4} \\ \frac{1}{8}+\frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} \\ -\frac{3}{4} \\ \frac{11}{24} \end{bmatrix} \Rightarrow$$

$$\mathbf{U} = \mathbf{K}^{-1}\mathbf{F}$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -\frac{7}{1152} \\ -\frac{23}{6912} \\ \frac{3}{128} \end{bmatrix} = \begin{bmatrix} -.00607 \\ -.00332 \\ .023434 \end{bmatrix} 371$$

Beam Example: Resultant nodal forces

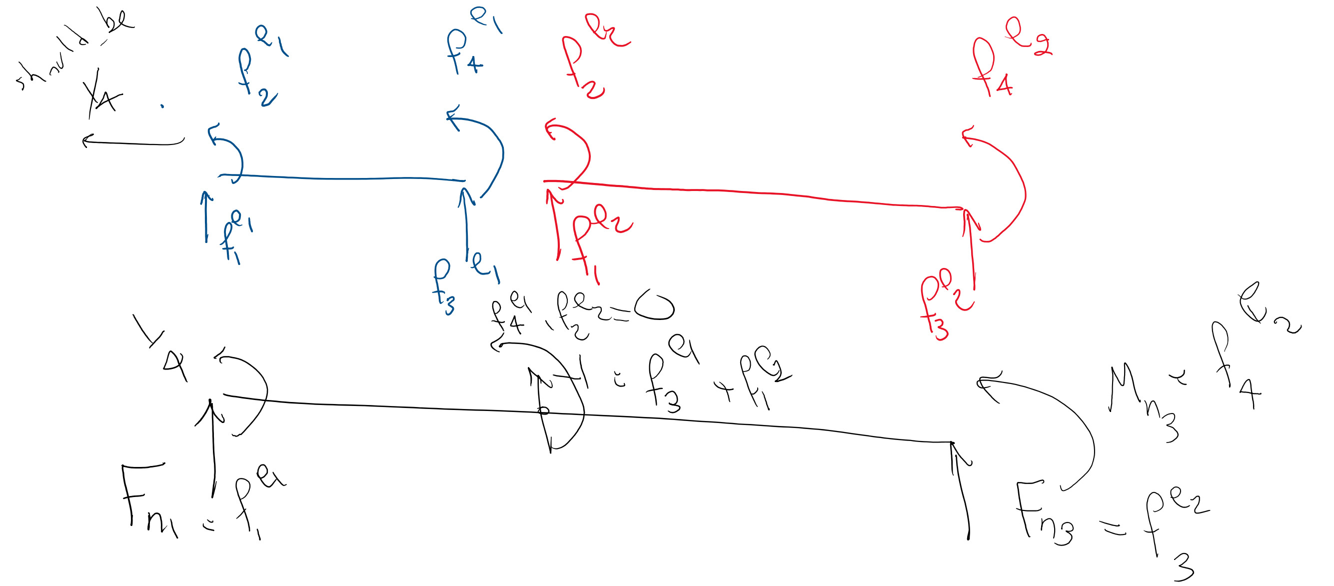


$M^{e_2} = [\bar{2}, 3, \bar{2}, \bar{3}]$

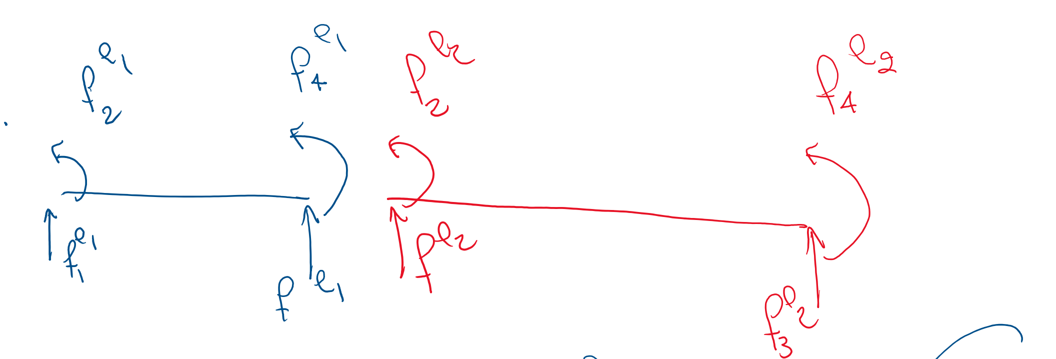
before slv. $a^{e_2} = [0, 0, \frac{1}{192}, 0]$

$a^{e_2} = [-.00332, .02334, \frac{1}{192}, 0]$

e	e_1	e_2
u^e	$\begin{bmatrix} 0 \\ -\frac{7}{1152} \\ -\frac{23}{6912} \\ \frac{3}{128} \end{bmatrix}$	$\begin{bmatrix} -\frac{23}{6912} \\ \frac{3}{128} \\ \frac{1}{192} \\ 0 \end{bmatrix}$



Basically, once we get the nodal forces, we can calculate the support forces:

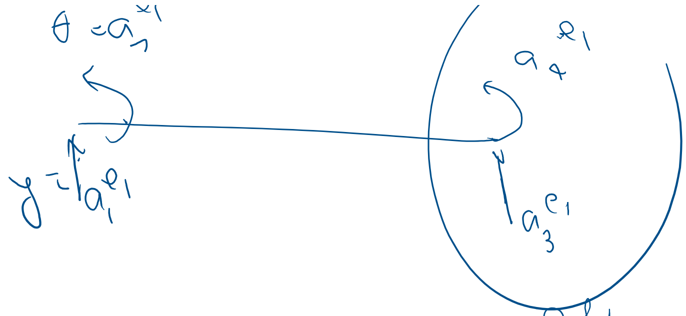


$a^{e_1} = \begin{bmatrix} 0 \\ -\frac{7}{1152} \\ -\frac{23}{} \end{bmatrix}$

$\theta = a_?^{e_1}$

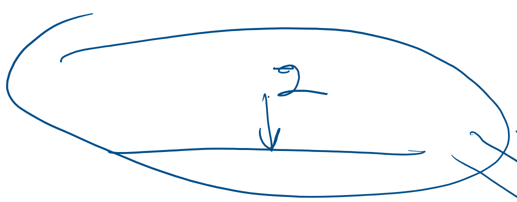
$a_4^{e_1}$

$$a^{e_1} = \begin{bmatrix} 0 \\ -\frac{7}{1152} \\ -\frac{23}{6912} \\ \frac{3}{128} \end{bmatrix} \rightarrow v^{e_1}$$



only from nodal displacement

$$\begin{bmatrix} f_1^{e_1} \\ f_2^{e_1} \\ \vdots \\ f_4^{e_1} \end{bmatrix} = K^{e_1} \begin{bmatrix} a_1^{e_1} \\ a_2^{e_1} \\ \vdots \\ a_4^{e_1} \end{bmatrix}$$

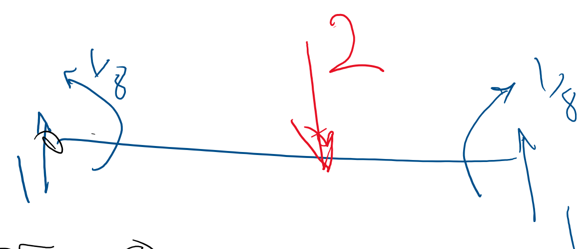


$$f_r^{e_2} = \begin{bmatrix} -1 \\ -1/8 \\ -1 \\ 1/8 \end{bmatrix}$$

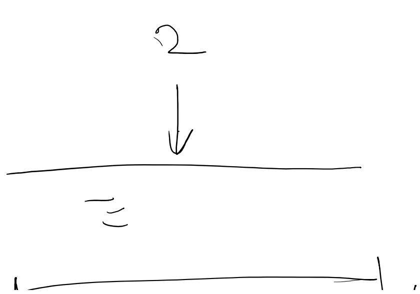
f_r

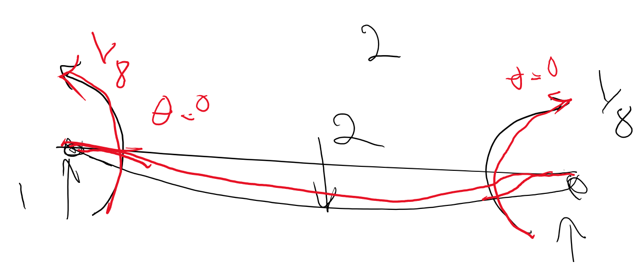
$-f_r^{e_2}$

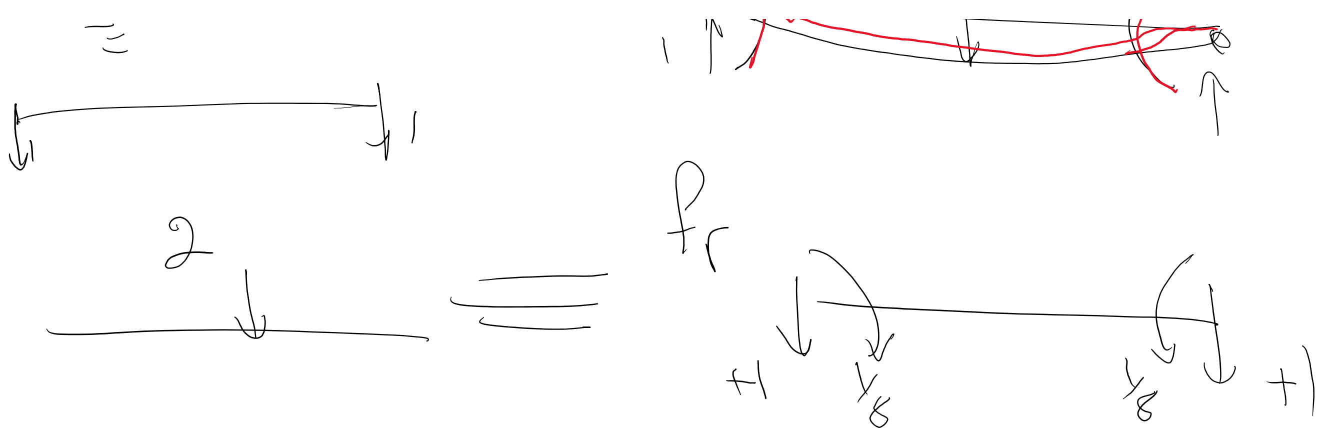
$-f_r^{e_1}$



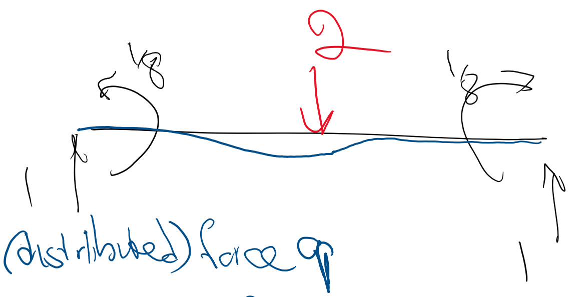
$$\Sigma F_y = 0$$
$$\Sigma M_1 = 0$$







$-f_r$

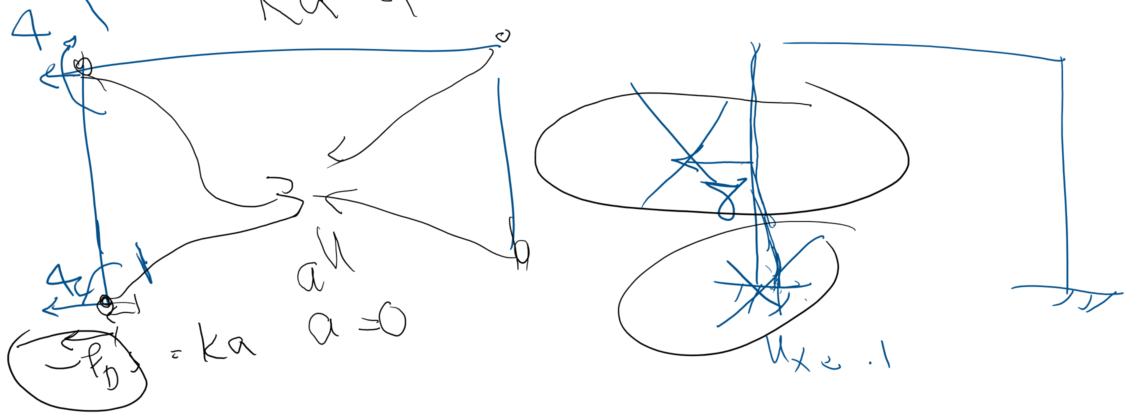


— $-f_r$ balances (distributed) force q

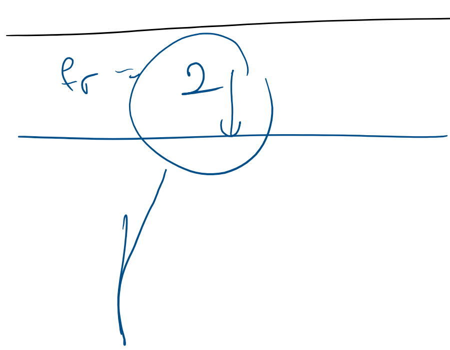
$-f_r$: makes all nodal a's zero

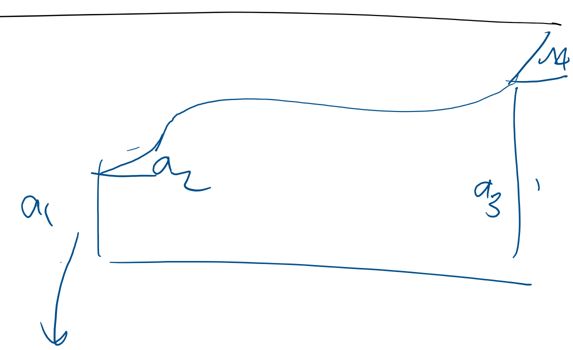
$Ka = F$

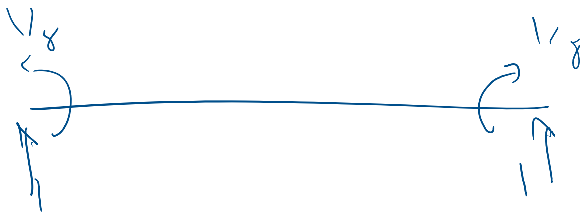
f_r



$$f_e = f_r - f_D$$







$$f_e = k_e a_e$$

$$-f_r$$

actual nodal force is $-f_r + \underbrace{ka}_{f_D}$

$$= -\underbrace{(f_r - f_D)}$$

more general case

$$\underbrace{f_e}_{\text{what we assembled to global force}} = \left(f_r + f_N - \overbrace{f_D}^{-ka}\right)$$

what we assembled to global force

$\underline{\underline{-f_e}}$ is acting on nodes of element

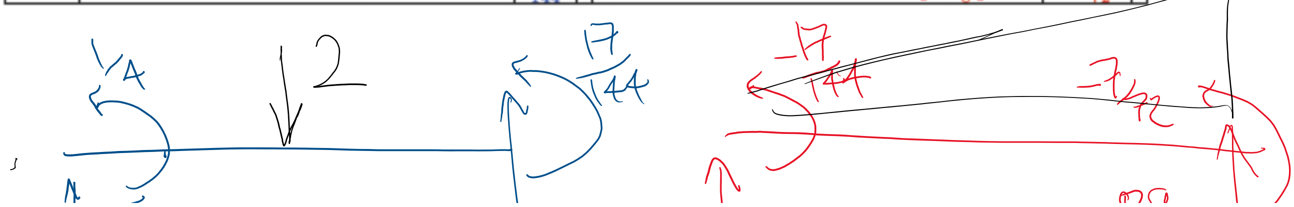
$$= -f_r - f_N + \underbrace{f_D}_{ka}$$

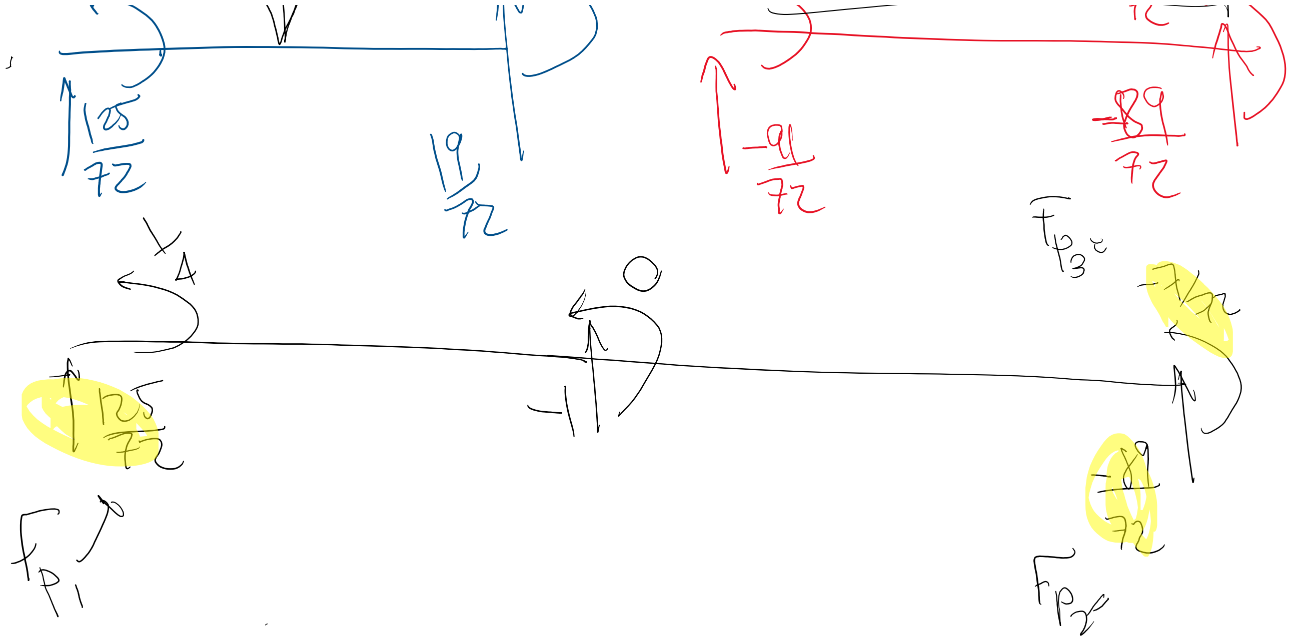
$$-f^e \qquad k^{e_1} a_1^e - f_r^{e_1} - \cancel{f_N^{e_1}} =$$

$$\begin{bmatrix} 96 & 24 & -96 & 24 \\ 24 & 8 & -24 & 4 \\ -96 & -24 & 96 & -24 \\ 24 & 4 & -24 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{7}{1152} \\ -\frac{23}{6912} \\ \frac{3}{128} \end{bmatrix} - \begin{bmatrix} -1 \\ -\frac{1}{8} \\ -1 \\ \frac{1}{8} \end{bmatrix} = \begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} \frac{125}{72} \\ \frac{1}{4} \\ \frac{19}{72} \\ \frac{17}{144} \end{bmatrix}$$

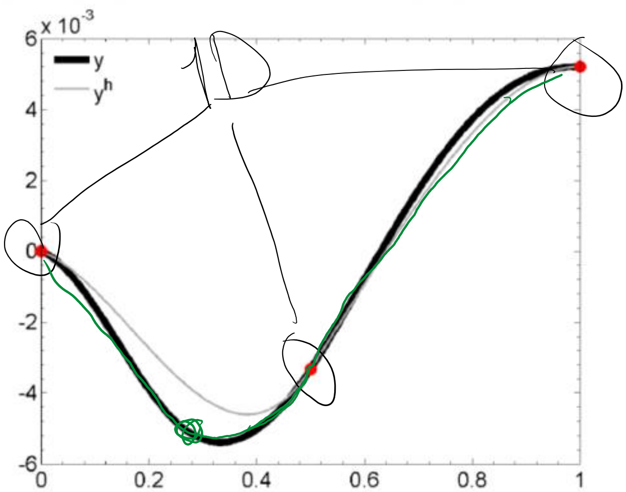
$$k^{e_2} a_2^e - f_r^{e_2} - \cancel{f_N^{e_2}} =$$

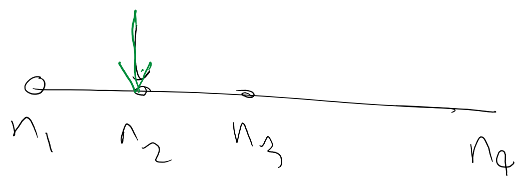
$$\begin{bmatrix} 192 & 48 & -192 & 48 \\ 48 & 16 & -48 & 8 \\ -192 & -48 & 192 & -48 \\ 48 & 8 & -48 & 16 \end{bmatrix} \begin{bmatrix} -\frac{23}{6912} \\ \frac{3}{128} \\ \frac{1}{192} \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{3}{4} \\ \frac{1}{12} \\ \frac{7}{4} \\ -\frac{1}{8} \end{bmatrix} = \begin{array}{c} 2 \\ 3 \\ 2 \\ 3 \end{array} \begin{bmatrix} -\frac{91}{72} \\ -\frac{17}{144} \\ -\frac{89}{72} \\ -\frac{7}{72} \end{bmatrix}$$





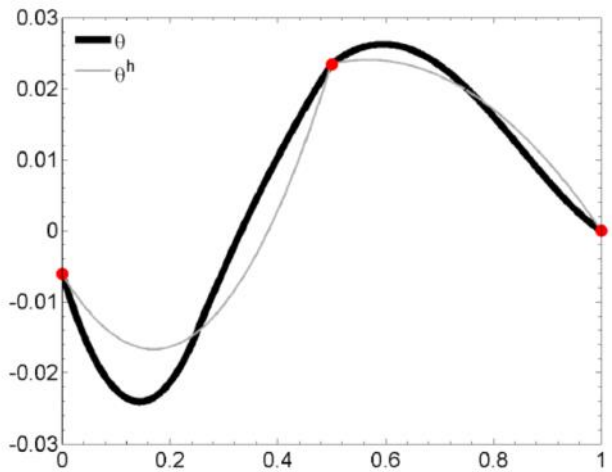
Beam Example: FEM vs. Exact solution: y





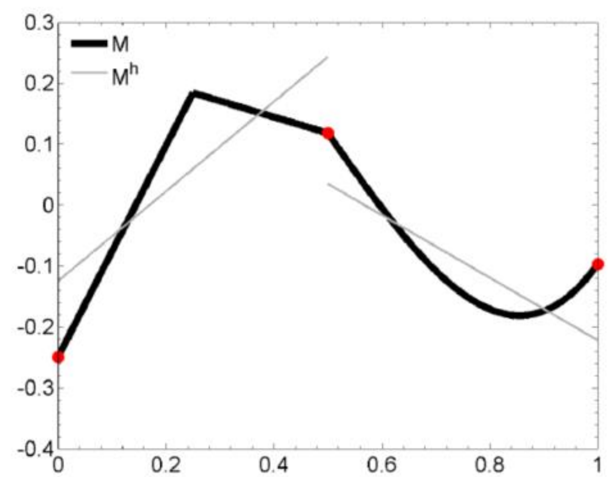
- Similar to bar element FEM and exact solutions match at nodes.
- This behavior is restricted to certain problems in 1D with constant material properties along the element and does not extend to more general cases.

Beam Example: FEM vs. Exact solution: θ



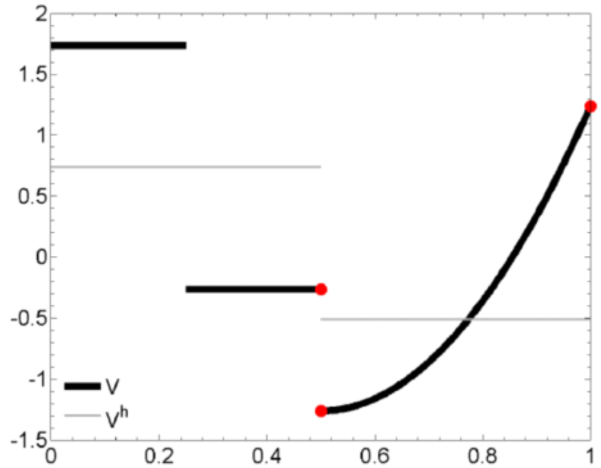
- Rotations at nodes (rotational dofs) match those from exact solution.
- Again we emphasize that while this behavior is shared for certain types of 1D problems, it does not extend to more general cases.

Beam Example: FEM vs. Exact solution: M



- It is clear that FEM solutions for M are much less accurate than those for y and θ when compared to exact solution.
- This is a general behavior where FEM accuracy decreases for solution derivatives.

Beam Example: FEM vs. Exact solution: V



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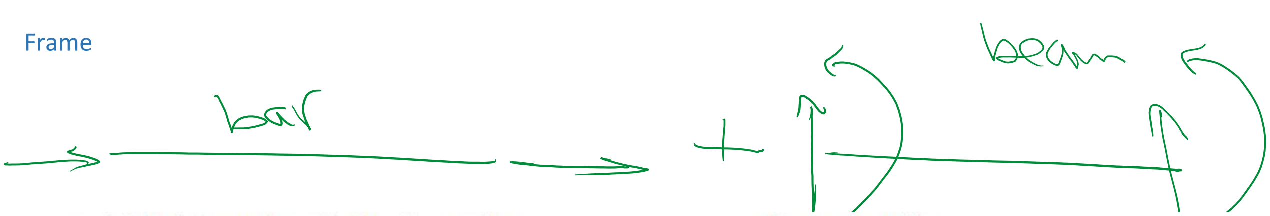
order of element

$$\|u - u^{exact}\|_2 = C h^{p+1}$$

$$\|u' - u^{exact'}\| = C h^{p+1-1}$$

$$\|u^{(s)} - u^{exact(s)}\| = C h^{p-s}$$

Frame



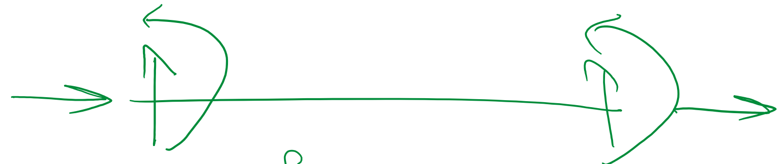
- Axial deformation (in local coord.)

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} = \begin{Bmatrix} f_{\bar{x}1} \\ f_{\bar{x}2} \end{Bmatrix}$$

- Beam bending

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \bar{v}_1 \\ \bar{\theta}_1 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} f_{\bar{y}1} \\ \bar{c}_1 \\ f_{\bar{y}2} \\ \bar{c}_2 \end{Bmatrix}$$

= frame

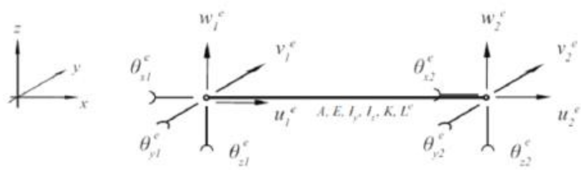


frame

- Element matrix equation (local coord.)

$$\begin{bmatrix} a_1 & 0 & 0 & -a_1 & 0 & 0 \\ 0 & 12a_2 & 6La_2 & 0 & -12a_2 & 6La_2 \\ 0 & 6La_2 & 4L^2a_2 & 0 & -6La_2 & 2L^2a_2 \\ -a_1 & 0 & 0 & a_1 & 0 & 0 \\ 0 & -12a_2 & -6La_2 & 0 & 12a_2 & -6La_2 \\ 0 & 6La_2 & 2L^2a_2 & 0 & -6La_2 & 4L^2a_2 \end{bmatrix} \begin{Bmatrix} \overline{u}_1 \\ \overline{v}_1 \\ \overline{\theta}_1 \\ \overline{u}_2 \\ \overline{v}_2 \\ \overline{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} \overline{f}_{x1} \\ \overline{f}_{y1} \\ \overline{c}_1 \\ \overline{f}_{x2} \\ \overline{f}_{y2} \\ \overline{c}_2 \end{Bmatrix} \qquad a_1 = \frac{EA}{L} \qquad a_2 = \frac{EI}{L^3}$$

3D frame element: local system



A 3D 12-freedom beam element defined in a local system

source: B. Torstenfelt; http://www.solid.iei.liu.se/Education/TMHL02/Book-Bars%20and%20Beams.pdf

- 12 local dofs are:

$$\mathbf{a}^{e\mathrm{T}} = \{u_1^e\ v_1^e\ w_1^e\ \theta_{x1}^e\ \theta_{y1}^e\ \theta_{z1}^e\ u_2^e\ v_2^e\ w_2^e\ \theta_{x2}^e\ \theta_{y2}^e\ \theta_{z2}^e\} \qquad (445)$$

- They correspond to axial and bending displacements, and torsional and bending rotations:

$u_1^e\ v_1^e\ w_1^e\ \theta_{x1}^e\ \theta_{y1}^e\ \theta_{z1}^e$	dofs of node n_1	(446a)
$u_2^e\ v_2^e\ w_2^e\ \theta_{x2}^e\ \theta_{y2}^e\ \theta_{z2}^e$	dofs of node n_2	(446b)
$\{u_1^e\ u_2^e\}$	axial displacements	(446c)
$\{v_1^e\ w_1^e\ v_2^e\ w_2^e\}$	vertical displacements	(446d)
$\{\theta_{x1}^e\ \theta_{x2}^e\}$	axial (torsional) rotation	(446e)
$\{\theta_{y1}^e\ \theta_{z1}^2\ \theta_{y2}^e\ \theta_{z2}^2\}$	bending rotations	(446f)

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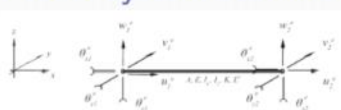
3D frame element: local system stiffness matrix



A 3D 12-freedom beam element defined in a local system

$$\boldsymbol{K}^e = \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & -a_1 & 0 & 0 & 0 & 0 & 0 \\ & b_1 & 0 & 0 & 0 & b_2 & 0 & -b_1 & 0 & 0 & 0 & b_2 \\ & & c_1 & 0 & -c_2 & 0 & 0 & 0 & -c_1 & 0 & -c_2 & 0 \\ & & & d_1 & 0 & 0 & 0 & 0 & 0 & -d_1 & 0 & 0 \\ & & & & c_3 & 0 & 0 & 0 & c_2 & 0 & c_4 & 0 \\ & & & & & b_3 & 0 & -b_2 & 0 & 0 & 0 & b_4 \\ & & & & & & a_1 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & b_1 & 0 & 0 & 0 & -b_2 \\ & & & & & & & & c_1 & 0 & c_2 & 0 \\ & & & & & & & & & d_1 & 0 & 0 \\ & s & y & m. & & & & & & & c_3 & 0 \\ & & & & & & & & & & & b_3 \end{bmatrix}$$

where

$a_1 = EA/L^e \qquad d_1 = GK/L^e$

$b_1 = 12EI_z/L^{e^3} \qquad b_2 = 6EI_z/L^{e^2} \qquad b_3 = 4EI_z/L^e \qquad b_4 = 2EI_z/L^e$

$c_1 = 12EI_y/L^{e^3} \qquad c_2 = 6EI_y/L^{e^2} \qquad c_3 = 4EI_y/L^e \qquad c_4 = 2EI_y/L^e$

source: B. Torstenfelt; http://www.solid.iei.liu.se/Education/TMHL02/Book-Bars%20and%20Beams.pdf

1. EA contributes to axial displacement.
2. EI_y and EI_z contributes to bending vertical displacement and rotations.
3. GK contributes to axial (torsional) rotation. G is the torsional modulus and K is torsional proportional constant for the cross section.

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3. GK contributes to axial (torsional) rotation. G is the torsional modulus and K is torsional proportional constant for the cross section.

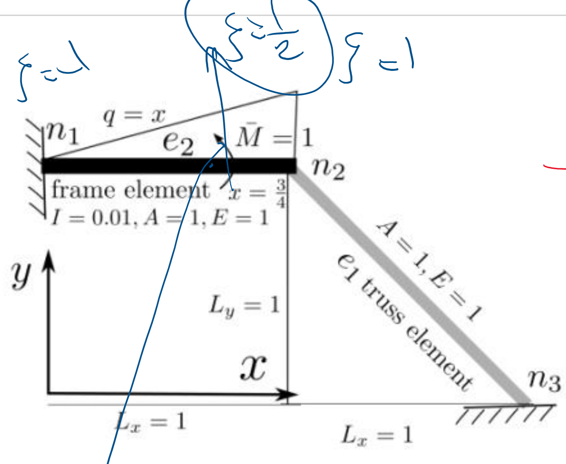
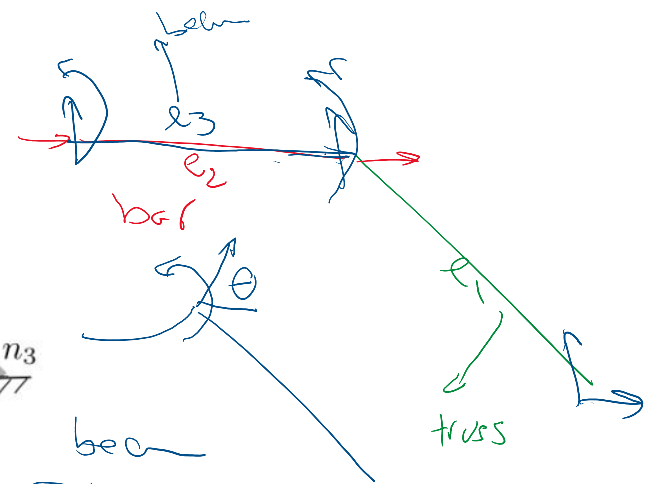


Figure 3: Frame and truss example.

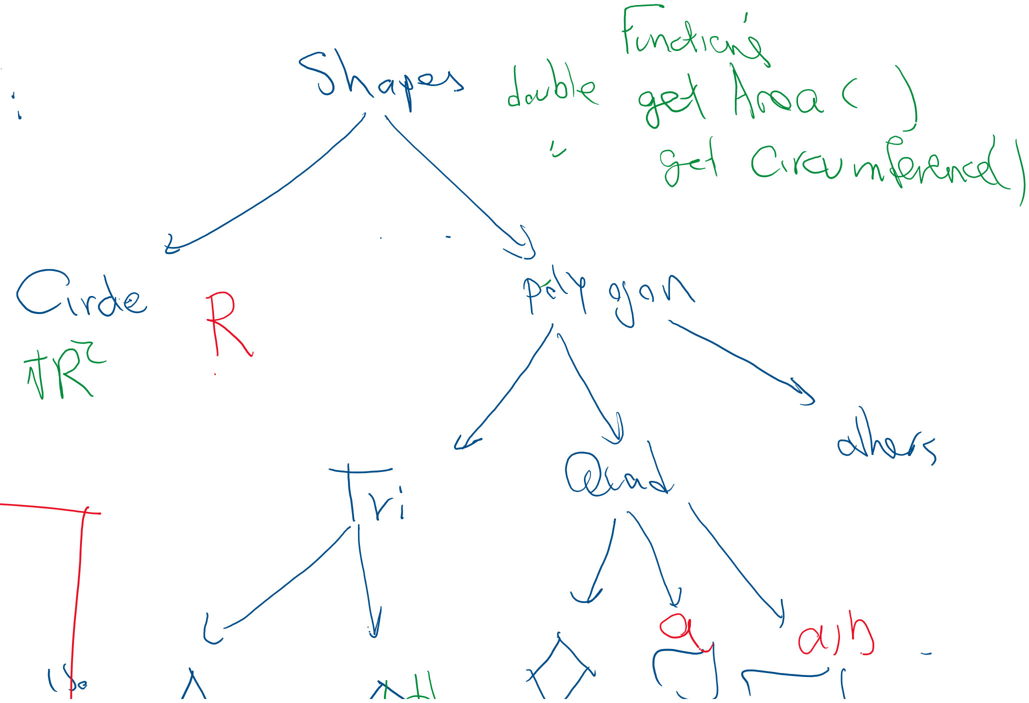


$$f_r^M = M \frac{d}{dx}\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}$$

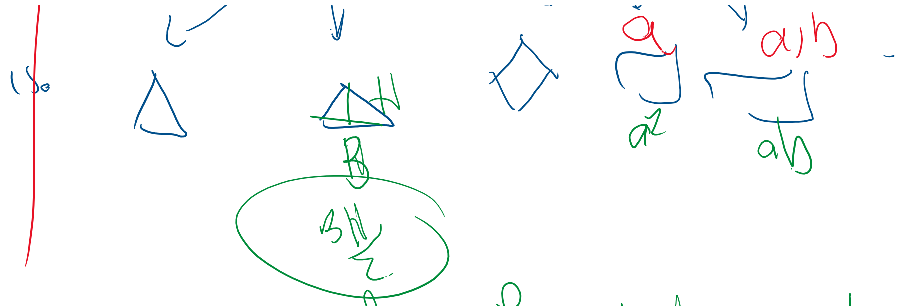
$$= M \frac{d}{d\xi}\begin{bmatrix} N_1 \\ \vdots \\ N_4 \end{bmatrix} \times \frac{1}{L/2}$$

Coding the FEM method

Class hierarchy :

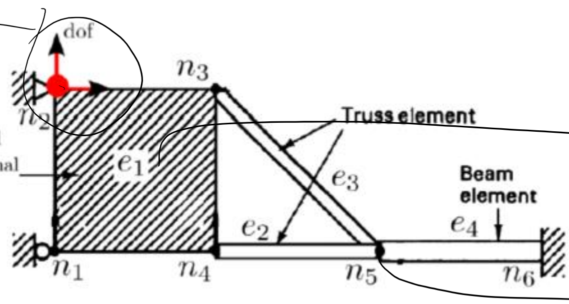


data



actual imp of get Area is done for those subclasses

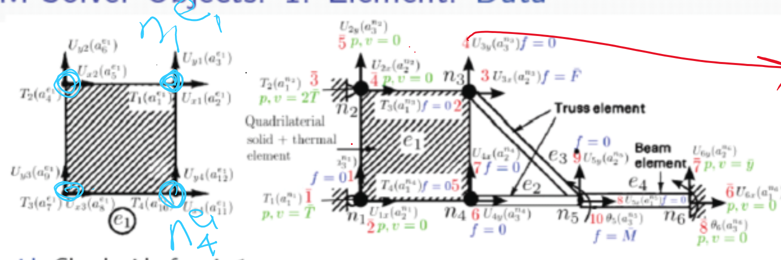
Some dofs at a node



element

node

FEM Solver Objects: 1. Element: Data



dof positions

LEM = [3, 2, 1, 4]

- id: Clearly, id of e_1 is 1.
- neNodes (n_n^e): Number of element nodes (*e.g.*, for element 1 $n_n^e = 4$).
- eNodes (LEM): Indices of element nodes in global system; *e.g.*, for e_1:

$$\mathrm{LEM}^{e_1} = \begin{bmatrix} 3 & 2 & 1 & 4 \end{bmatrix}$$

- number of element dof (nedof: n_f^e): can be different than n_n^e; for $e_1 : n_f^e = 12$.
- edofs ($\mathbf{a}^e$): n_f^e vector of dofs; *e.g.*, for e_1:

$$\begin{aligned} a_1^e &= \begin{bmatrix} a_1^{e1} & a_2^{e1} & \cdots & a_{n_f^e}^{e1} \end{bmatrix} \\ &= \begin{bmatrix} T_1 & U_{x1} & U_{y1} & T_2 & U_{x2} & U_{y2} & T_3 & U_{x3} & U_{y3} & T_4 & U_{x4} & U_{y4} \end{bmatrix} \end{aligned}$$

- dofMap ($\mathbf{M}_t^e$): map from element to global dofs; *e.g.*, for e_i we will observe:

$$\mathbf{M}^{e_1}{}_t = \begin{bmatrix} 2 & 3 & 4 & \bar{3} & \bar{4} & \bar{5} & \bar{1} & \bar{2} & 1 & 5 & 6 & 7 \end{bmatrix}$$

$\rightarrow k^e \rightarrow K$

$f^e \rightarrow F$

- stiffness matrix (ke: $\mathbf{k}^e$): $n^e_f \times n^e_f$ local stiffness (conductivity, *etc.*) matrix.
- force vectors (fde, foe, fee: $\mathbf{f}^e_D, \mathbf{f}^e_o, \mathbf{f}^e_e$): A variety of element n^e_f load vectors such as essential BC ($\mathbf{f}^e_D$), sum of other forces ($\mathbf{f}^e_o = \mathbf{f}^e_r + \mathbf{f}^e_N + \cdots$, *etc.*), and element total $\mathbf{f}^e_e = \mathbf{f}^e_o - \mathbf{f}^e_D$.
- {physics}: Physics represented by the element; *e.g.*, for e_1 physics are solid and thermal.
- eType: one or more than one object that identify the element type. Examples are shape and order of element (linear triangle, *etc.*).

we don't need these

class PhyElement
{

```
int id;
int neNodes; // # element nodes
vector<int> eNodes; // element node vector
vector <PhyNode*> eNodePtrs;
int nedof; // # element dof
VECTOR edofs; // element dofs
vector<int> dofMap;
ElementType eType;
int matID;
MATRIX ke; // element stiffness matrix
VECTOR foe; // element force vector from all sources other than essential BC
VECTOR fde; // element essential BC force
VECTOR fee; // all element forces
```

Functions:

- Calculate forces
- Calculate stiffness
- Assemble

bar

Virtual

$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ truss $\frac{AE}{L}\begin{bmatrix} k_b & -k_b \\ -k_b & k_b \end{bmatrix}$

normal

the same