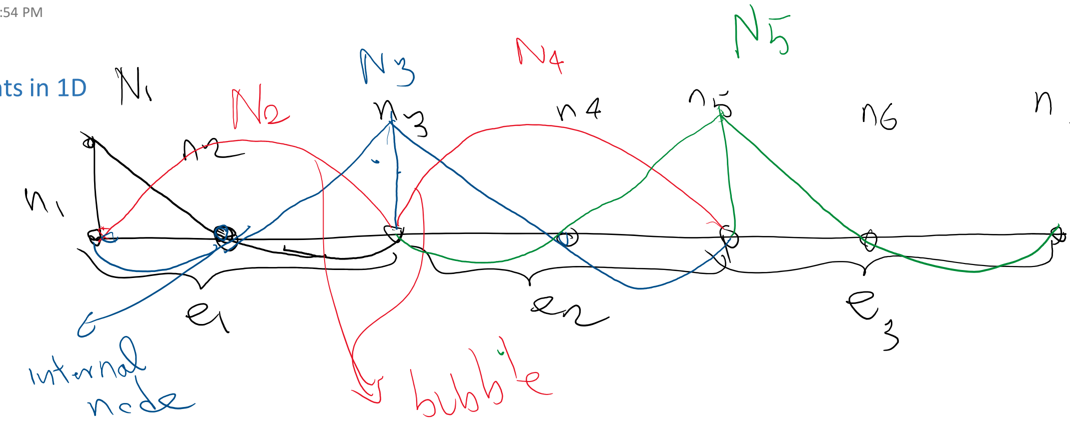
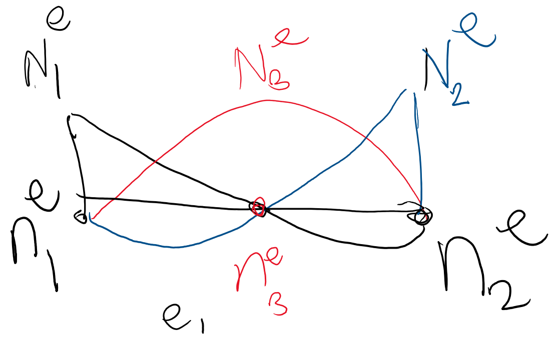
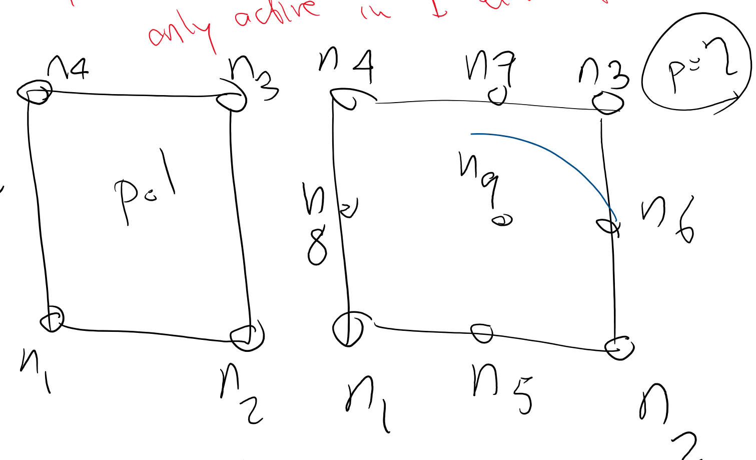
FEM20241125

Monday, November 25, 2024 12:54 PM

Higher order elements in 1D

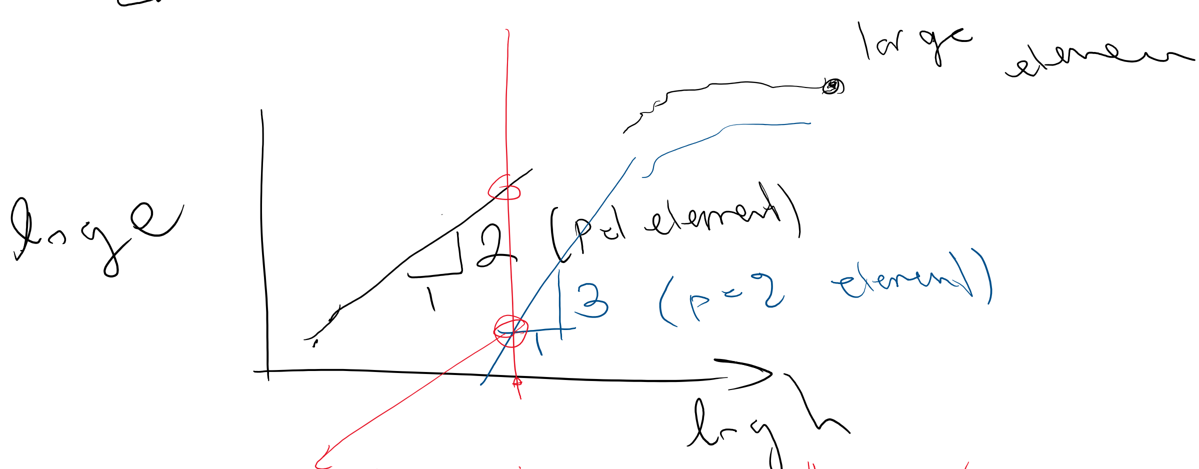




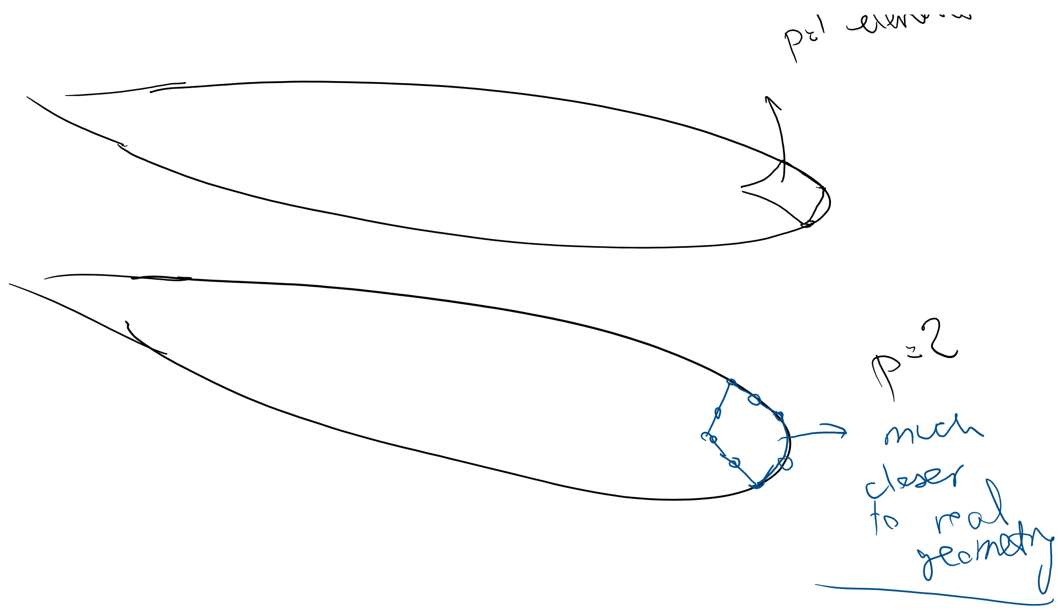


$$e = \| u^h - u^{exct} \| = C h^{P+1}$$

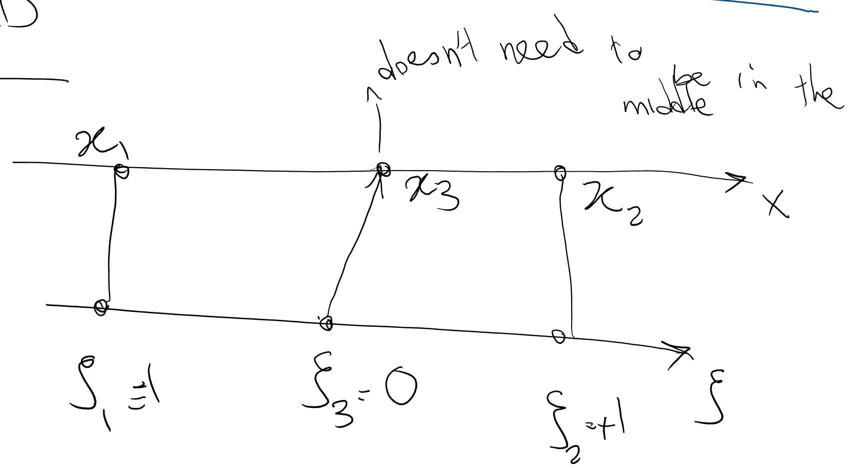
FEM for smooth enough exact solution



p=2 gives much smaller error



p=2 element in 1D



we write the shape functions in ξ

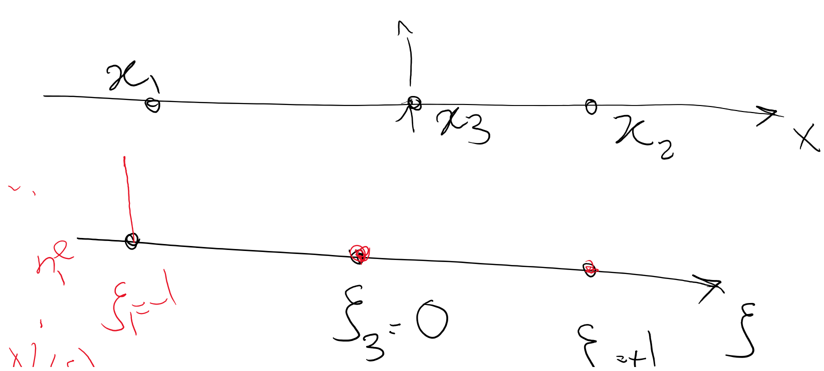
$$N(\xi) = [N_1(\xi), N_2(\xi), N_3(\xi)]$$

$x(\xi)$ is need
$$\begin{cases} x(\xi_1) = x_1 \\ x(\xi_2) = x_2 \\ x(\xi_3) = x_3 \end{cases}$$

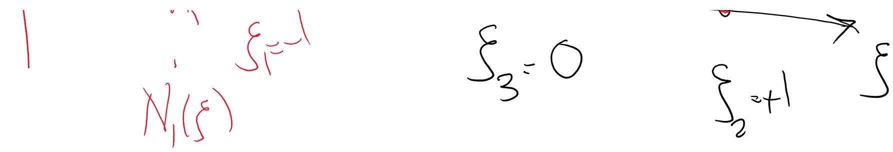
Shape functions in ξ.

$$N_1(\xi) = \alpha_0 + \alpha_1\xi + \alpha_2\xi^2 + \alpha_3\xi^3 + \dots$$

3 eqns



3 eqns



$$\left.\begin{array}{l} N_1(-1) = 1 \\ N_1(0) = 0 \\ N_1(1) = 0 \end{array}\right\} \rightarrow \begin{array}{l} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{array}$$

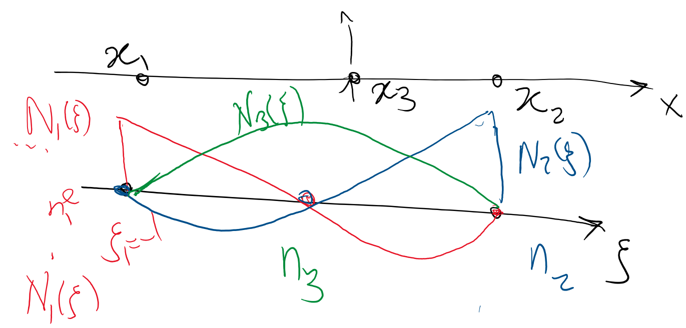
$$N_1(\xi) = L_1(\xi) = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)}$$

Lagrange polynomials

$$N_1(\xi_1) = \frac{(\xi_1 - \xi_2)(\xi_1 - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} = 1 \qquad N_1(\xi_2) = \frac{(\xi_2 - \xi_2)(\xi_2 - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} = 0$$

$$N_1(\xi_3) = \frac{(\xi_3 - \xi_2)(\xi_3 - \xi_3)}{\text{denom}} = 0$$

$$N_1(\xi) = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} =$$

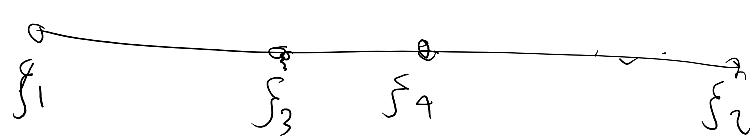


$$\frac{(\xi - 1)(\xi - 0)}{(-1 - 1)(-1 - 0)} = \frac{\xi(\xi - 1)}{2}$$

$$N_2(\xi) = L_2(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_3)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)} = \frac{(\xi - (-1))(\xi - 0)}{(1 - (-1))(1 - 0)} = \frac{\xi(\xi + 1)}{2}$$

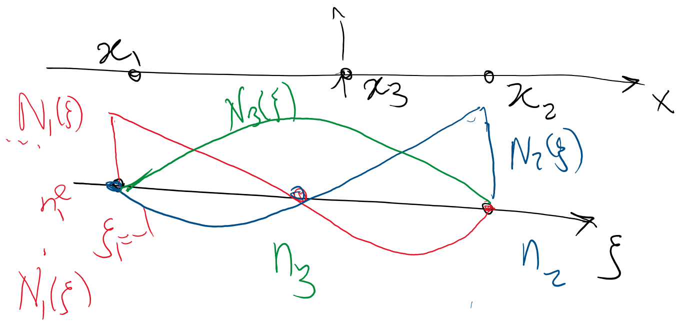
$$N_3(\xi) = L_3(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} = \frac{(\xi - (-1))(\xi - 1)}{(0 - (-1))(0 - 1)} = 1 - \xi^2$$

$$N_i(\xi) = \frac{\prod_{j=1, j \neq i}^{n} (\xi - \xi_j)}{\prod_{j=1, j \neq i}^{n} (\xi_i - \xi_j)}$$



$$N(\xi) = [N_1, N_2, N_3]$$

$$= \left[\frac{\xi(\xi-1)}{2}, \frac{\xi(\xi+1)}{2}, 1-\xi^2 \right]$$



bar problem

$$\text{WK: } \int_e w' \underbrace{EA}_{D^e} \underset{\sim}{u'}\, dx \longrightarrow k^e = \int_e B^{e^T} D^e B^e\, dx$$

$$L_m = (\ )'$$

$$B^e = \frac{d}{dx} N^e(\xi) = \underbrace{\left(\frac{dN^e}{d\xi}\right)}_{B^e_\xi} \cdot \left(\frac{d\xi}{dx}\right) \longrightarrow \cancel{\xi(x)}$$

we never have this for higher order element

OR

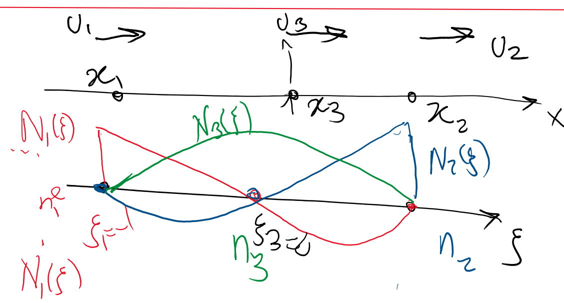
$$\boxed{x(\xi)} : \frac{d\xi}{dx} = \frac{1}{\frac{dx}{d\xi}}$$

very easy to get this

Goal

$$x(\xi) \longrightarrow J = \frac{dx}{d\xi}$$

$$B = B_\xi \frac{1}{J}$$



$$x(\xi) = ?$$

solution $u(x)$

$$u(x) = u_1 N_1(\xi) + u_2 N_2(\xi) + u_3 N_3(\xi)$$

$$u(\xi_1) = u_1 \underbrace{N_1(\xi_1)}_{1} + u_2 \underbrace{N_2(\xi_1)}_{0} + u_3 \underbrace{N_3(\xi_1)}_{0}$$

$$= u_1$$

$$\boxed{u(\xi_i) = u_i} \quad i = 1, 2, 3$$

$$x(\xi_1) = x_1$$
$$x(\xi_2) = x_2$$
$$x(\xi_3) = x_3$$

$$x(\xi) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \alpha_3 \xi^3 \ldots$$

find $\alpha_0, \alpha_1, \alpha_2$

BORING!

SOLUTION

$$x(\xi) = x_1 N_1(\xi) + x_2 N_2(\xi) + x_3 N_3(\xi)$$

$$x(\xi) = x_1 N_1(\xi) + x_2 N_2(\xi) + x_3 N_3(\xi)$$

$$x(\xi_3) = x_1 \underbrace{N_1(\xi_3)}_{0} + x_2 \underbrace{N_2(\xi)}_{0} \quad x_3 \underbrace{N_3(\xi_3)}_{1} = x_3$$

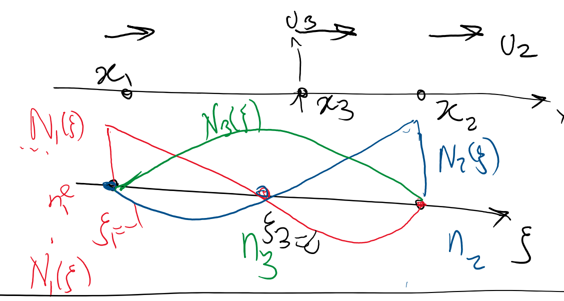
$$\boxed{x(\xi_i) = x_i \quad i = 1,2,3}$$

GEOMETRY

$\underbrace{\text{ISO}}$PARAMETRIC element

geometry (x) & solution u are interpolated with the same shape functions

$$k^e = \int_e B^{e\,T} D^e B^e \, dx$$



$$B^e = B^e_\xi \frac{1}{J} \qquad B^e_\xi = \frac{d}{d\xi} N$$

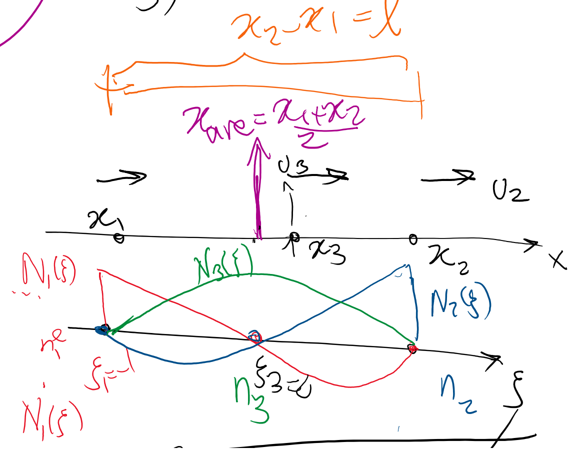
$$\boxed{J(\xi) = \frac{dx(\xi)}{d\xi}}$$

I want to calculate this

$$\boxed{x(\xi) = N_1 x_1 + N_2 x_2 + N_3 x_3 = \frac{\xi(\xi-1)}{2} x_1 + \frac{\xi(\xi+1)}{2} x_2 + (1-\xi^2) x_3}$$

$$J(\xi) = \frac{dx(\xi)}{d\xi} = \left(\xi - \frac{1}{2}\right) x_1 + \left(\xi + \frac{1}{2}\right) x_2 - 2\xi x_3$$

$$= \left[\frac{x_2 - x_1}{2}\right] + 2\xi\left(\frac{x_1 + x_2}{2} - x_3\right)$$

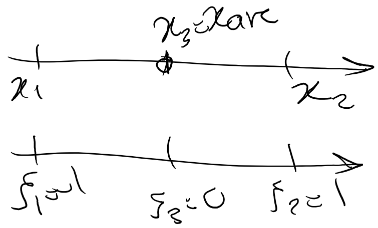


$$J(\xi) = \frac{l^e}{2} + 2\xi(x_{ave} - x_3) > 0$$

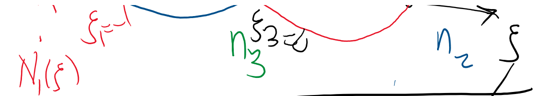
may be one point

special case

$x_3 = x_{ave}$



$$J(\xi) = \frac{l^e}{2}$$



one pond

(2)

(1) $$K^e = \int_e (B^e)^t D^e B^e \, dx$$

$$B^e = B^e_\xi \frac{1}{J}, \quad B^e_\xi = \frac{d}{d\xi} N$$

$$D^e = EA$$

(3) $$dx = \frac{dx}{d\xi} d\xi, \quad J = \frac{dx}{d\xi}$$

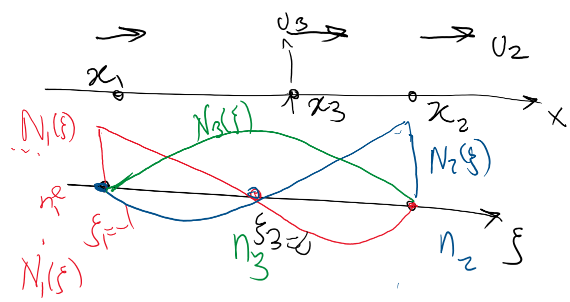
$$B_\xi = \frac{d}{d\xi} N = \frac{d}{d\xi}[N_1, N_2, N_3] = \frac{d}{d\xi}\left[\frac{-\xi+\xi^2}{2}, \frac{\xi+\xi^2}{2}, 1-\xi^2\right]$$

$$\Rightarrow \boxed{B_\xi = \left[\xi - \frac{1}{2}, \xi + \frac{1}{2}, -2\xi\right]} \quad (4)$$

Plug 2, 3, 4 in (1)

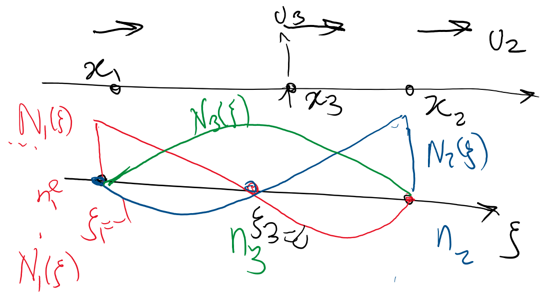
$$K^e = \int_{\xi=-1}^{\xi=1} (B^e)^t EA (B^e) \overbrace{(J \, d\xi)}^{dx}$$

$$B^e = \frac{1}{J} B^e_\xi$$



$$K^e = \int_{-1}^{1} \frac{1}{J} (B^e_\xi)^t EA \frac{1}{\cancel{J}} B^e \cancel{J} \, d\xi = \underline{\int_{\xi=-1}^{1} \frac{1}{J} (B^e_\xi)^t (B^e_\xi) \, d\xi}$$

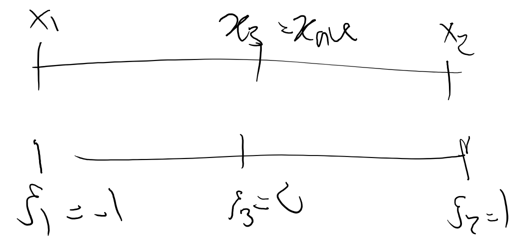
$$k^e = \int_{\xi=-1}^{1} \underbrace{\frac{1}{J(\xi)} \begin{bmatrix} \xi-\frac{1}{2} \\ \xi+\frac{1}{2} \\ -2\xi \end{bmatrix} \begin{bmatrix} \xi-\frac{1}{2} & \xi+\frac{1}{2} & -2\xi \end{bmatrix}}_{I_{3\times3}(\xi)} d\xi$$



(5)

$$J(\xi) = \frac{l_e}{2} + 2\xi(x_{ave} - x_3)$$

Simple case $x_3 = x_{ave}$



$$J(\xi) = \frac{l_e}{2}$$

$$I_{12}(\xi) = \frac{(\xi-\frac{1}{2})(\xi+\frac{1}{2})}{\frac{l_e}{2} + 2\xi(x_{ave} - x_3)}$$

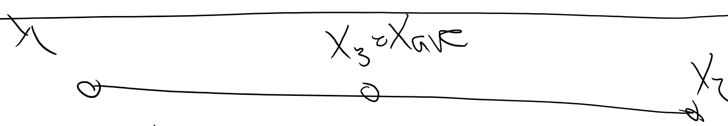
rational function

$X_3 \neq X_{ave}$ Very difficult

$X_3 = X_{ave}$

$$I_{12} = \underbrace{\frac{2}{l_e}(\xi-\tfrac{1}{2})(\xi+\tfrac{1}{2})}_{\text{polynomial}}$$

we can easily integrate this



$$k^e = \frac{AE}{L} \begin{bmatrix} 7/3 & -8/3 & 1/3 \\ & 7/3 & -8/3 \\ \text{sym} & & 16/3 \end{bmatrix}$$

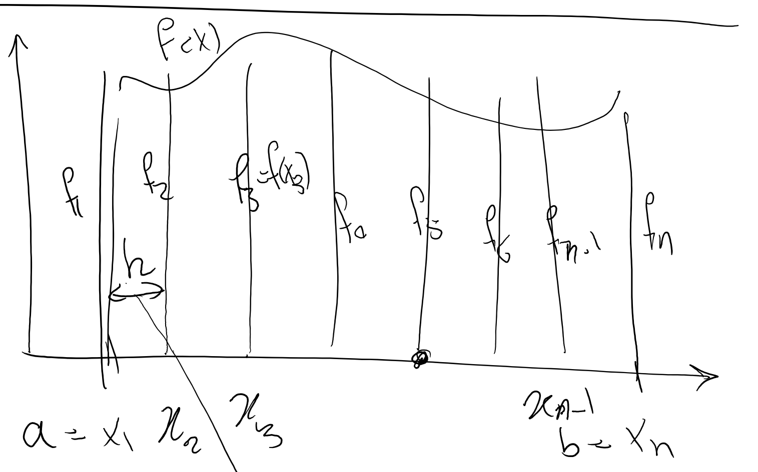
$x_3 \neq x_{ave} \Longrightarrow$ need to use

Numerical Integration (quadrature)

Numerical Integration (Quadrature)

Newton-Cotes method

$$\int_a^b f(x)\,dx = L\Big(\sum_{i=1}^{n} \underbrace{w_i}_{\text{quadrature weights}} f(\underbrace{x_i}_{\text{quadrature pts}})\Big)$$



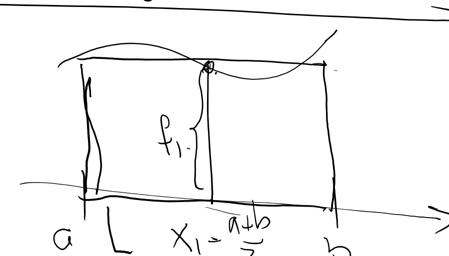
$n-1$ equal size segment

$h = \dfrac{L}{n-1}$, $\underbrace{L = b - a}_{\text{length of the segment}}$

Example 1 $n=1$

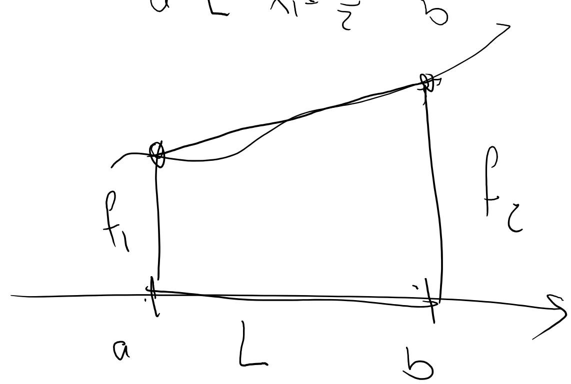
$$\int_a^b f(x)dx = L f_1 = L(\underbrace{w_1}_{1} f_1)$$

square integration rule



$$\int_a^b f(x)dx = L\left(\frac{f_1 + f_2}{2}\right)$$

$$= L\left(\frac{1}{2} f_1 + \frac{1}{2} f_2\right)$$



$$= L\left(\underbrace{\frac{1}{2}A_1}_{\omega_1} + \underbrace{\frac{1}{2}A_2}_{\omega_2}\right)$$