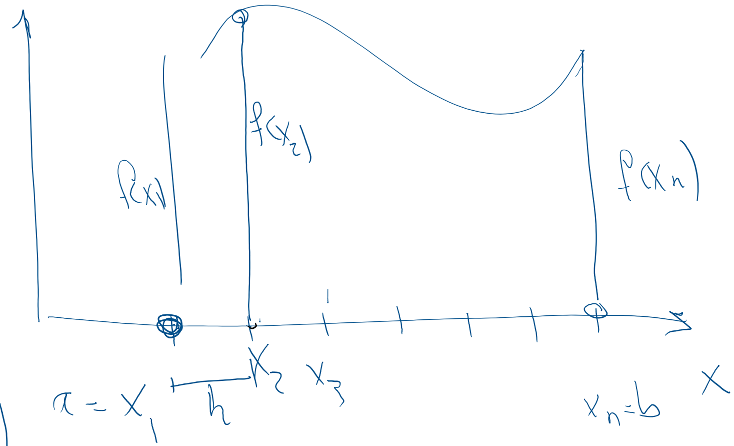
FEM20241202

Monday, December 2, 2024 12:57 PM

Continue: Newton-Cotes

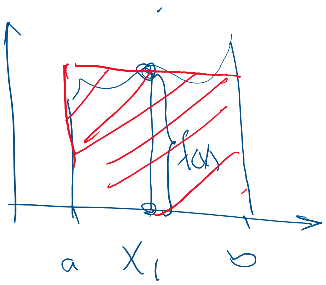
$$\int_a^b f(x)\,dx \approx \underbrace{(b-a)}_{\text{length of domain}} \left(\sum \omega_i f(x_i)\right)$$



n : points

n-1 segments $\quad h = \dfrac{b-a}{n-1}$

1 point: rectangle scheme



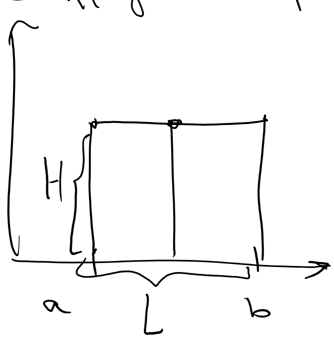
$$\int_a^b f(x)dx \approx (b-a)\ f(x_1) \quad (b-a) \times \omega_1 f(x_1)$$

for $\omega_1 = 1$

Does this rule integrate polynomials exactly?

$O = 0$ ✓

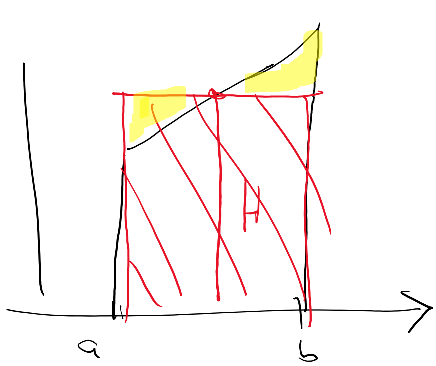
"Oh"



$(b-a)\,f(x_1) = L\,H$ exact?

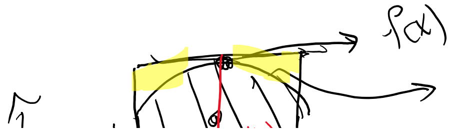
$O = 1$ ✓

max polynomial order



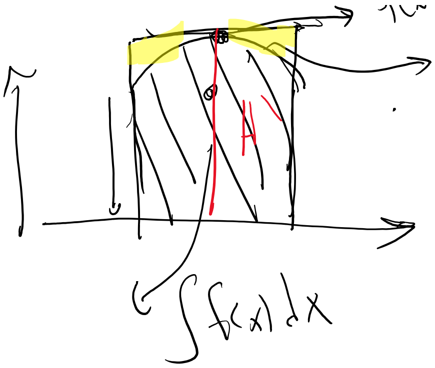
$\int f(x) = LH$?

$O = 2$



rectangle rule adds the highlighted region

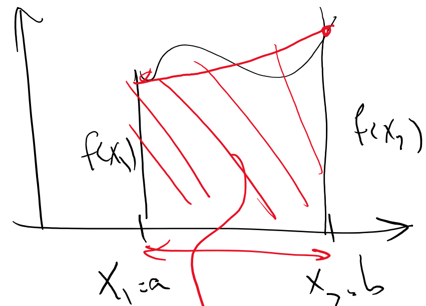
$o = 2$



rectangle rule adds the highlighted region

$$\int_a^b f(x)\,dx \neq (b-a)\, f(x_1)$$

Trapezoidal rule



$$\int_a^b f(x)\,dx = (b-a)\left(w_1 f(x_1) + w_2 f(x_2)\right)$$

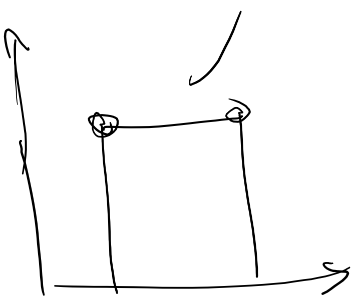
$w_1 = ?\ \ w_2 = ?$

$$= (b-a)\left(\frac{1}{2} f(x_1) + \frac{1}{2} f(x_2)\right)$$

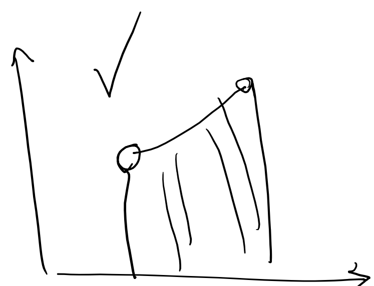
$w_1 = \frac{1}{2}$ $\quad w_2 = \frac{1}{2}$

$$\left(\frac{f(x_1) + f(x_2)}{2}\right)(b-a)$$

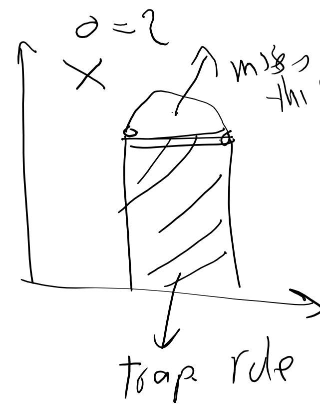
$o = \phi$



$o = 1$



$o = 2$



Trapezoidal scheme is not good as with one added point, still integrates only up to order = 1 exactly.

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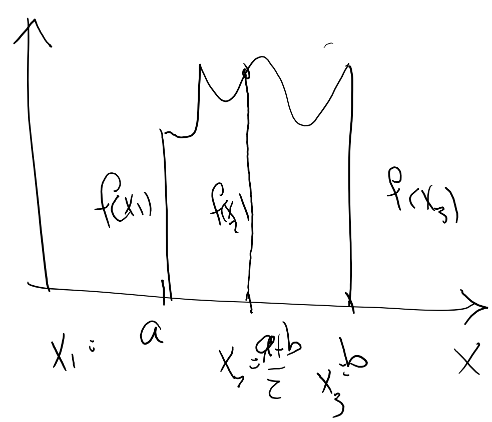
3 point scheme (Simpson's scheme)

$$\int_a^b f(x)\,dx \approx (b-a)\left(\omega_1 f(x_1) + \omega_2 f(x_2) + \omega_3 f(x_3)\right)$$

$\omega_1 = \frac{1}{6}$, $x_1 = a$; $\omega_2 = \frac{4}{6}$, $x_2 = \frac{a+b}{2}$; $\omega_3 = \frac{1}{6}$, $x_3 = b$



3-unknowns

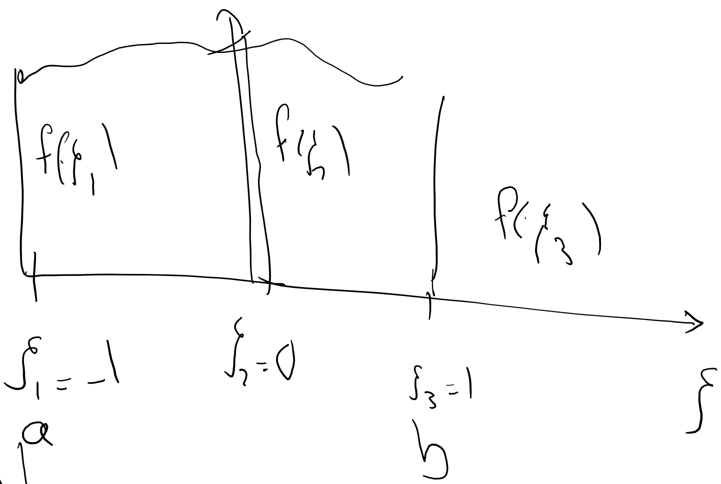
$\omega_1, \omega_2, \omega_3 = ?$

$\omega_1, \omega_2, \omega_3 = ?$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$\underbrace{a_0 + a_1 x + a_2 x^2}_{3 \text{ eqns}}$

Derive $\omega_1, \omega_2, \omega_3$



$$\int_{-1}^{1} f(\xi)\,d\xi = \left(1 - (-1)\right)\left(\omega_1 f(\xi_1) + \omega_2 f(\xi_2) + \omega_3 f(\xi_3)\right)$$

$$\boxed{\int_{-1}^{1} f(\xi)\,d\xi = 2\left(\omega_1 f(-1) + \omega_2 f(0) + \omega_3 f(1)\right)} \quad \star$$

we want $\star$ to be exact for $f = 1$, $f = \xi$, $f = \xi^2$

i) $f = 1$: $\quad 2 = 2(\omega_1 + \omega_2 + \omega_3) \rightarrow$

ii) $f = \xi$: $\quad \int_{-1}^{1} \xi\,d\xi = \frac{\xi^2}{2}\Big|_{-1}^{1} = 0 = 2\left(\omega_1(-1) + \omega_2(0) + \omega_3(1)\right) \rightarrow$

iii) $f = \xi^2$: $\quad \int_{-1}^{1} \xi^2\,d\xi = \frac{\xi^3}{3}\Big|_{-1}^{1} = \frac{2}{3} = 2\left(\omega_1(-1)^2 + \omega_2(0)^2 + \omega_3(1)^2\right) \rightarrow$

$$\begin{cases} \omega_1 + \omega_2 + \omega_3 = 1 & \text{i} \\ -\omega_1 + \omega_3 = 0 & \text{ii} \\ \omega_1 + \omega_3 = \frac{1}{3} & \text{iii} \end{cases}$$

ii $\rightarrow \omega_1 = \omega_3 \rightarrow$ plug in iii $\quad \omega_1 = \omega_3 = \frac{1}{6} \rightarrow$ plug in

i $\rightarrow \quad \omega_2 = \frac{4}{6} = \frac{2}{3}$

$$\boxed{\int_a^b f(x)\,dx \approx (b-a)\Big(\underset{1/6}{\omega_1}\, f(\underset{a}{x_1}) + \underset{4/6}{\omega_2}\, f(\underset{\frac{a+b}{2}}{x_2}) + \underset{1/6}{\omega_3}\, f(\underset{b}{x_3})\Big)}$$

Simpson's rule

$o = 0$ ✓

$o = 1$ ✓

$o = 2$ ✓

$o = 3$ ✓ yes we'll get this for free

$o = 4$ ✗

Simpson's rules with odd number of points give one extra polynomial order integration

$\alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \dots$

Newton Cotes scheme with

points

n

exact order of integral

$$o = \begin{cases} n-1 & n \text{ even} \\ n-1+1 & n \text{ odd} \end{cases}$$

n	name	o
1	rect.	1
2	trap.	1
3	simp.	3
4		3

better

better

$$o = \begin{cases} n-1 & n \text{ even} \\ n & n \text{ odd} \end{cases}$$

TABLE 5.5 *Newton-Cotes numbers and error estimates*

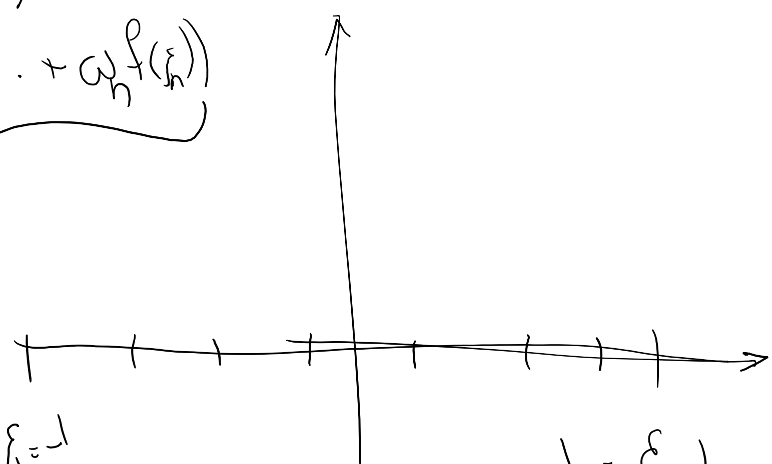
npts	Number of intervals n	C_0^n	C_1^n	C_2^n	C_3^n	C_4^n	C_5^n	C_6^n	Upper bound on error R_n as a function of the derivative of F
2	1	$\frac{1}{2}$	$\frac{1}{2}$						$10^{-1}(b-a)^3F^{II}(r)$
3	2	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$					$10^{-3}(b-a)^5F^{IV}(r)$
4	3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$				$10^{-3}(b-a)^5F^{IV}(r)$
5	4	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$			$10^{-6}(b-a)^7F^{VI}(r)$
6	5	$\frac{19}{288}$	$\frac{75}{288}$	$\frac{50}{288}$	$\frac{50}{288}$	$\frac{75}{288}$	$\frac{19}{288}$		$10^{-6}(b-a)^7F^{VI}(r)$
7	6	$\frac{41}{840}$	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$	$\frac{27}{840}$	$\frac{216}{840}$	$\frac{41}{840}$	$10^{-9}(b-a)^9F^{VIII}(r)$

Always use odd # point schemes

Trick in getting $\omega_1, \omega_2, \ldots$

$$\int_{-1}^{1} f(\xi)\,d\xi = (1-(-1))\underbrace{\left(\omega_1 f(\xi_1) + \ldots + \omega_n f(\xi_n)\right)}$$

in 1 step find ω_i



$a = \xi_1 = -1$ $\quad b = \xi_n = 1$

if

$$f = L_i(\xi) = \frac{\prod_{j=1, \neq i}^{n} (\xi - \xi_j)}{\prod_{j=1, \neq i}^{n} (\xi_i - \xi_j)}$$

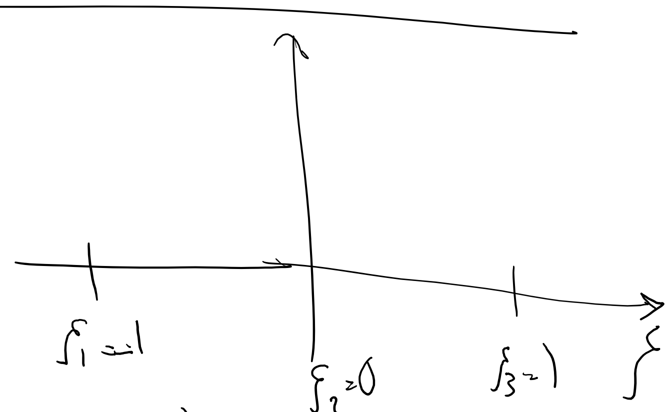
$$\int_{-1}^{1} L_i(\xi)\,d\xi = 2\left(\omega_1 \underbrace{L_i(\xi_1)}_{0} + \ldots + \omega_{i-1}\underbrace{L_i(\xi_{i-1})}_{0} + \omega_i \underbrace{L_i(\xi_i)}_{1} + \ldots\right)$$

$$\boxed{\omega_i = \frac{\int_{-1}^{1} L_i(\xi)\,d\xi}{2}}$$

Example

Example

$$\omega_2 = \frac{\int_{-1}^{1} L_2(\xi)\,d\xi}{2}$$



$$L_2(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)} = \frac{(\xi-(-1))(\xi-1)}{(0-(-1))(0-1)} = 1-\xi^2$$

$$\omega_2 = \frac{\int_{-1}^{1}(1-\xi^2)\,d\xi}{2} = \frac{2-2/3}{2} = \frac{4}{6}$$

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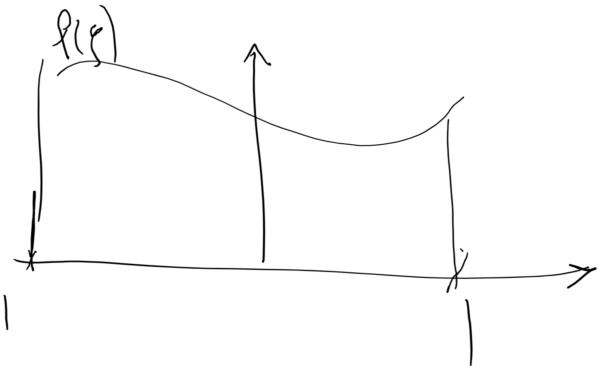
2 pt scheme

Gauss Quadrature:

$$\int_{-1}^{1} f(\xi)\,d\xi = \cancel{(1-(-1))}\left(\omega_1 f(\xi_1) + \omega_2 f(\xi_2)\right)$$



\# unknowns $\omega_1, \omega_2, \xi_1, \xi_2$

$\xi_1 \neq -1$? $\xi_2 \neq 1$?

$$f(\xi) = \underset{①}{\alpha_0} + \underset{②}{\alpha_1 \xi} + \underset{③}{\alpha_2 \xi^2} + \underset{④}{\alpha_3 \xi^3} \;\Big|\; + \alpha_4 \xi^4 \cdots$$

Finding $\omega_1, \omega_2, \xi_1, \xi_2$

$$\int f(\xi)\,d\xi = \left(\omega_1 f(\xi_1) + \omega_2 f(\xi_2)\right)\cancel{(2)}$$

$$\int_{-1}^{1} 1\,d\xi = 2 = \omega_1 \times 1 + \omega_2 \times 1 \quad ①$$

$$\int_{-1}^{1} \xi\,d\xi = \frac{\xi^2}{2}\Big|_{-1}^{1} = 0 = \omega_1 \xi_1 + \omega_2 \xi_2 = 0 \quad ②$$

4 nonlinear eqns

$$\int_{-1}^{1}\xi\, d\xi = \frac{\xi^2}{2}\Big|_{-1}^{1} = 0 = \omega_1\xi_1 + \omega_2\xi_2 = 0 \quad ②$$

$$\int_{-1}^{1}\xi^2\, d\xi = \frac{\xi^3}{3}\Big|_{-1}^{1} = \frac{2}{3} = \omega_1\xi_1^2 + \omega_2\xi_2^2 = 2/3 \quad ③$$

$$\int_{-1}^{1}\xi^3\, d\xi = \frac{\xi^4}{4}\Big|_{-1}^{1} = 0 = \omega_1\xi_1^3 + \omega_2\xi_2^3 = 0 \quad ④$$

4 unknowns

② multiply by ξ_1^2 :

④

$$\begin{cases} \omega_1\xi_1^3 + \omega_2\xi_1^2\xi_2 = 0 \\ \omega_1\xi_1^3 + \omega_2\xi_2^3 = 0 \end{cases}$$

subtract

$$\omega_2\xi_2(\xi_2^2 - \xi_1^2) = 0 \quad : \quad \omega_2\xi_2(\xi_2 - \xi_1)(\xi_2 + \xi_1) = 0$$

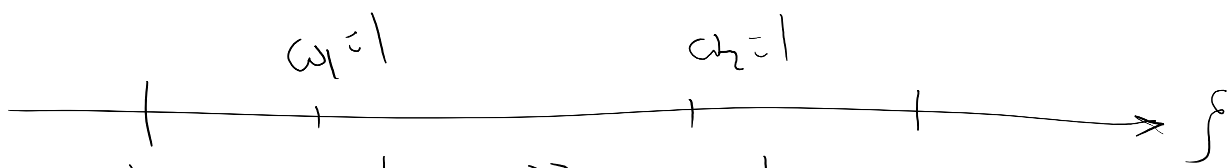
$$\rightarrow \begin{cases} \omega_2 = 0 \quad \times \ 1pt & \text{or} \\ \xi_2 = 0 \longrightarrow \text{Dead end} & \text{or} \\ \xi_2 - \xi_1 = 0 \quad 1pt & \text{or} \\ \xi_2 + \xi_1 = 0 \end{cases}$$

$\longrightarrow \xi_1 = -\xi_2 \rightarrow$ plug in ② $\omega_1\xi_1^2 + \omega_2\xi_2^2 = \frac{2}{3}$

$$\begin{rcases} \overbrace{(\omega_1 + \omega_2)}^{2}(\xi_2)^2 = \frac{2}{3} \\ \text{eq1} \quad \omega_1 + \omega_2 = 2 \end{rcases} \longrightarrow \xi_2^2 = \frac{1}{3} \longrightarrow \xi_2 = \frac{1}{\sqrt{3}}$$

$$\rightarrow \xi_1 = -\xi_2 = \frac{-1}{\sqrt{3}}$$

$$\int_{-1}^{1} f(\xi)\,d\xi = \omega_1 f(\xi_1) + \omega_2 f(\xi_2)$$



$\xi_1 = \frac{-1}{\sqrt{3}} \approx -.577..$ $\quad \xi_2 = \frac{1}{\sqrt{3}} = +.577..$

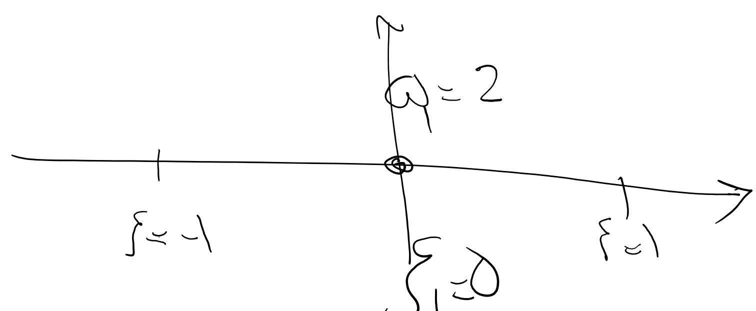
What polynomial it integrates exactly $\quad 3$

$\alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \alpha_3 \xi^3$

2 pt Gauss Quad.

TABLE 5.6 *Sampling points and weights in Gauss-Legendre numerical integration (interval −1 to +1)*

n	r_i			α_i		
1	0.	(15 zeros)		2.	(15 zeros)	
2	±0.57735	02691	89626	1.00000	00000	00000
3	±0.77459	66692	41483	0.55555	55555	55556
	0.00000	00000	00000	0.88888	88888	88889
4	±0.86113	63115	94053	0.34785	48451	37454
	±0.33998	10435	84856	0.65214	51548	62546
5	±0.90617	98459	38664	0.23692	68850	56189
	±0.53846	93101	05683	0.47862	86704	99366
	0.00000	00000	00000	0.56888	88888	88889
6	±0.93246	95142	03152	0.17132	44923	79170
	±0.66120	93864	66265	0.36076	15730	48139
	±0.23861	91860	83197	0.46791	39345	72691



$$\int_{-1}^{1} f(\xi)\,d\xi = \omega_1 f(\xi_1) = 2 f(0)$$

Rectangle $\quad$ order 1

$\omega_1 = 1 \quad \xi_1 = -.57735026\ldots = \frac{-1}{\sqrt{3}}$

$\omega_2 = 1 \quad \xi_2 = +.577\ldots = \frac{1}{\sqrt{3}}$

order = $2 \times 2 - 1 = 3$

$n = 3 \quad \omega_1 = 0.5555\ldots56 \quad \xi_1 = -.77459666\ldots$

$\omega_2 = 0.88888 \quad \xi_2 = 0$

$\omega_3 = 0.55\ldots56 \quad \xi_3 = +77459666\ldots$

$\xi_1 = -\sqrt{\frac{3}{5}}$ $\quad w_1 = \frac{5}{9}$

$\xi_2 = 0$ $\quad \omega_2 = \frac{8}{9}$

$\xi_3 = \sqrt{\frac{3}{5}}$ $\quad \omega_3 = \frac{5}{9}$

Roots of this are

n	$P_n(x)$
0	1 $\to 0$
1	x
2	$\frac{1}{2}(3x^2-1)$ $\to -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$
3	$\frac{1}{2}(5x^3-3x)$ $\to \mp\sqrt{\frac{3}{5}}, 0$
4	$\frac{1}{8}(35x^4-30x^2+3)$
5	$\frac{1}{8}(63x^5-70x^3+15x)$
6	$\frac{1}{16}(231x^6-315x^4+105x^2-5)$
7	$\frac{1}{16}(429x^7-693x^5+315x^3-35x)$
8	$\frac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35)$
9	$\frac{1}{128}(12155x^9-25740x^7+18018x^5-4620x^3+315x)$
10	$\frac{1}{256}(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63)$

Figure 4: Legendre polynomials (Source: http://en.wikipedia.org/wiki/Legendre_polynomials)

Gauss point Locations

Once ξ_i's are known

$$\boxed{W_i = \int_{-1}^{1} L_i(\xi)\, d\xi \quad \text{for Gauss quadrature}}$$

TABLE 5.6 *Sampling points and weights in Gauss-Legendre numerical integration (interval −1 to +1)*

n	r_i			α_i		
1	0.	(15 zeros)		2.	(15 zeros)	
2	±0.57735	02691	89626	1.00000	00000	00000
3	±0.77459	66692	41483	0.55555	55555	55556
	0.00000	00000	00000	0.88888	88888	88889
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3	±0.77459 66692 41483	0.55555 55555 55556
	0.00000 00000 00000	0.88888 88888 88889
4	±0.86113 63115 94053	0.34785 48451 37454
	±0.33998 10435 84856	0.65214 51548 62546
5	±0.90617 98459 38664	0.23692 68850 56189
	±0.53846 93101 05683	0.47862 86704 99366
	0.00000 00000 00000	0.56888 88888 88889
6	±0.93246 95142 03152	0.17132 44923 79170
	±0.66120 93864 66265	0.36076 15730 48139
	±0.23861 91860 83197	0.46791 39345 72691

How do we use in practice

1. **50 Points** Use a 3 point Gauss and 5 point Newton-Cotes quadrature rule to evaluate the following integral and obtain their respective errors with respect to exact value of the integral $I_e = \tan^{-1}(2) - \tan^{-1}(-1)$. Quadrature points and weights are given in fig. 1.

$$I = \int_{-1}^{2} \frac{\mathrm{d}x}{1+x^2}$$

$f(x) = \frac{1}{1+x^2}$

HW7

$n = 5$

$h = \frac{b-a}{n-1} = \frac{3}{4}$

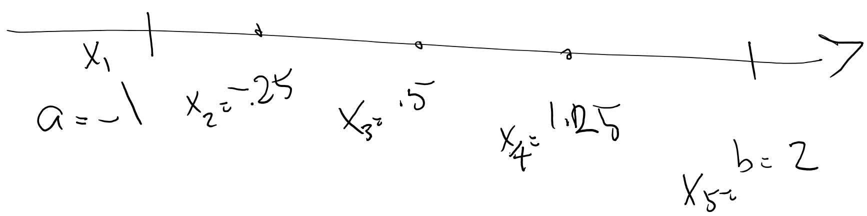

TABLE 5.5 *Newton-Cotes numbers and error estimates*

n pt's	Number of intervals $n-1$	C_0^n	C_1^n	C_2^n	C_3^n	C_4^n	C_5^n	C_6^n	Upper bound on error R_n as a function of the derivative of F
2	1	$\frac{1}{2}$	$\frac{1}{2}$						$10^{-1}(b-a)^3F^{II}(r)$
3	2	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$					$10^{-3}(b-a)^5F^{IV}(r)$
4	3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$				$10^{-3}(b-a)^5F^{IV}(r)$
5	4	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$			$10^{-6}(b-a)^7F^{VI}(r)$
	5	$\frac{19}{288}$	$\frac{75}{288}$	$\frac{50}{288}$	$\frac{50}{288}$	$\frac{75}{288}$	$\frac{19}{288}$		$10^{-6}(b-a)^7F^{VI}(r)$
	6	$\frac{41}{840}$	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$	$\frac{27}{840}$	$\frac{216}{840}$	$\frac{41}{840}$	$10^{-9}(b-a)^9F^{VIII}(r)$

$\int f(x)dx \approx$

$\frac{7}{90} f(x_1) + \frac{32}{90} f(x_2) + \frac{12}{90} f(x_3)$

$+ \frac{32}{90} f(x_4) + \frac{7}{90} f(x_5)$

Gauss Quadrature

-1 2

Gauss Quadrature

$$I = \int_{-1}^{2} \frac{1}{1+x^2}\,dx$$

$x \rightarrow \xi$

$$x = \overset{x_1}{(-1)}\underbrace{\left(\frac{1-\xi}{2}\right)}_{N_1(\xi)} + \overset{x_2}{2}\underbrace{\left(\frac{1+\xi}{2}\right)}_{N_2(\xi)}$$

$$x(\xi) = \frac{\xi - 1 + 2 + 2\xi}{2} = \frac{3\xi + 1}{2}$$

$$\Downarrow \quad dx = \frac{3}{2}\,d\xi$$

$$I = \int_{-1}^{1} \underbrace{\left[\left(\frac{1}{1+x(\xi)^2}\right)\frac{3}{2}\right]}_{\text{Integrand } g(\xi)} d\xi = \omega_1 g(\xi_1) + \omega_2 g(\xi_2) + \omega_3 g(\xi_3)$$

Apply Gauss Quad

$\omega_1 = \frac{5}{9} \quad \xi_1 = -\sqrt{\frac{3}{5}}$

$\omega_2 = \frac{8}{9} \quad \xi_2 = 0$

$\omega_3 = \frac{5}{9} \quad \xi_3 = \sqrt{\frac{3}{5}}$