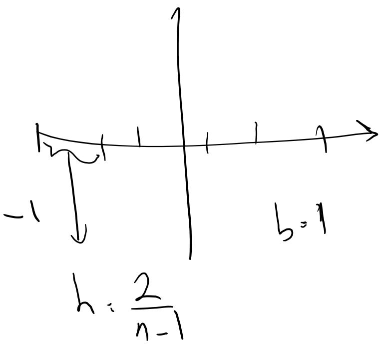
FEM20241203

Tuesday, December 3, 2024 12:55 PM

$f(\xi) = \alpha_0 + \alpha_1 \xi + \dots + \alpha_o \xi^{o}$ → order of polynomial

NC $\int_{-1}^{1} f(\xi)\,d\xi = (1 - -1)\left(\sum_{i=1}^{n} \omega_i f(\xi_i)\right)$

equi-distant



GQ $\int_{-1}^{1} f(\xi)\,d\xi \; \cancel{(2)} \sum_{i=1}^{n} w_i f(\xi_i)$

$n \to o$

$n \longrightarrow o$

NC $\quad o = \begin{cases} n-1 & \text{even } n \\ n & \text{odd } n \end{cases}$

GC $\quad o = 2n-1$ →

each GP is worth twice NC point

$o \to n$

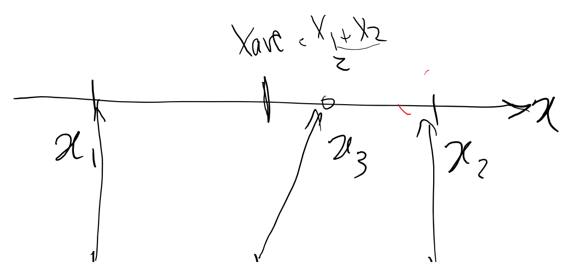
$o \longrightarrow n$

NC $\quad n = \begin{cases} o & o \text{ odd} \\ o+1 & o \text{ even} \end{cases}$ → often the case

G $\quad n = \text{ceil}\left(\frac{o+1}{2}\right)$

Application of this:

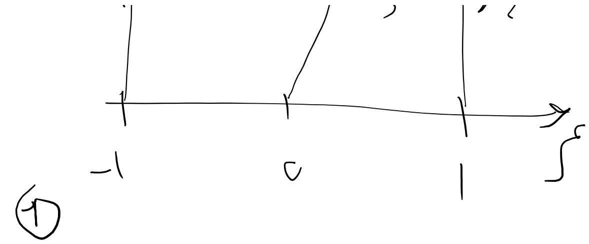
$k_{ij}^e = \int^{x_3} B^e D B^e \, dx \qquad T_{ave}(\xi)$



$$k^e = \underbrace{\int_{x_1}^{x_3} B^e D B^e \, dx}_{I_{3\times3}(\xi)}$$

$$= \int_{-1}^{1} \frac{EA(\xi)}{J(\xi)} \begin{bmatrix} \xi-\frac{1}{2} \\ \xi+\frac{1}{2} \\ -2\xi \end{bmatrix} \begin{bmatrix} \xi-\frac{1}{2} & \xi+\frac{1}{2} & -2\xi \end{bmatrix} d\xi$$

$$J(\xi) = 2\xi(x_{ave}-x_3) + \frac{l_e}{2}$$



- E & A are constant
  - E → homog.
  - A → prismatic cross sect. AE
- unskewed element $x_3 = x_{ave} \Rightarrow J_e = \frac{l_e}{2}$

} ⇒ integrand I is a polynomial

$$I_{12}(\xi) = \frac{EA}{l_e/2}\left(\xi-\frac{1}{2}\right)\left(\xi+\frac{1}{2}\right)$$ 2nd order polynomial

Maximum order over all i, j is $\underline{\underline{2}}$

$\boxed{\text{order} = 2}$

→ NC $\quad n_{NC} = o+1 = \underline{\underline{3}}$ Simpson's rule

→ GQ $\quad n_{GQ} = \text{ceil}\left(\frac{o+1}{2}\right) = \text{ceil}\left(\frac{3}{2}\right) = \underline{2}$

Integrating k^e with NC

TABLE 5.5 *Newton-Cotes numbers and error estimates*

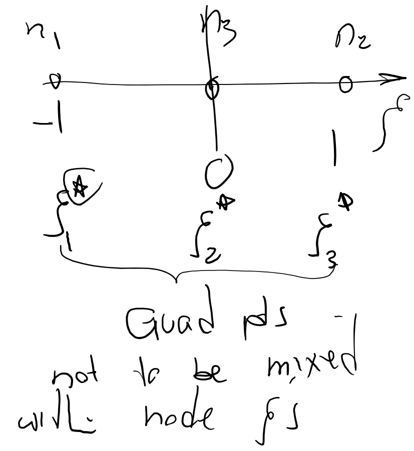
n pts	Number of intervals n	C_0^n	C_1^n	C_2^n	C_3^n	C_4^n	C_5^n	C_6^n	Upper bound on error R_n as a function of the derivative of F
2	1	$\frac{1}{2}$	$\frac{1}{2}$						$10^{-1}(b-a)^3F^{II}(r)$
3	2	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$					$10^{-3}(b-a)^5F^{IV}(r)$
4	3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$				$10^{-3}(b-a)^5F^{IV}(r)$
5	4	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$			$10^{-6}(b-a)^7F^{VI}(r)$
6	5	$\frac{19}{288}$	$\frac{75}{288}$	$\frac{50}{288}$	$\frac{50}{288}$	$\frac{75}{288}$	$\frac{19}{288}$		$10^{-6}(b-a)^7F^{VI}(r)$
7	6	$\frac{41}{840}$	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$	$\frac{27}{840}$	$\frac{216}{840}$	$\frac{41}{840}$	$10^{-9}(b-a)^9F^{VIII}(r)$

Simpson's rule with 3 pts

6	$\frac{41}{840}$	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$	$\frac{27}{840}$	$\frac{216}{840}$	$\frac{41}{840}$	$10^{-9}(b-a)^9 F^{\text{VIII}}(r)$

7

$$k^e = \int_{a=-1}^{b=1} \underline{I}_{3\times3}(\xi)\,d\xi = \underbrace{(\underset{1}{b} - \underset{-1}{a})}_{\text{need this in NC}} \left(\frac{1}{6}\underline{I}_{3\times3}(\xi_1^*) + \frac{4}{6}\underline{I}_{3\times3}(\xi_2^*) + \frac{1}{6}\underline{I}_{3\times3}(\xi_3^*)\right) =$$

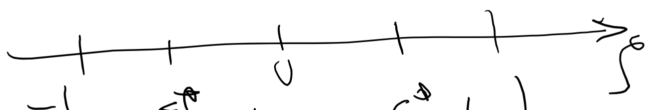


$$\boxed{k^e = \frac{1}{3}\left(\underline{I}_{3\times3}(-1) + 4\,\underline{I}_{3\times3}(0) + \underline{I}_{3\times3}(1)\right) \qquad \underline{I}_{3\times3}(\xi) = \frac{EA(\xi)}{J(\xi)} \begin{bmatrix} \xi-\frac{1}{2} \\ \xi+\frac{1}{2} \\ -2\xi \end{bmatrix} \begin{bmatrix} \xi-\frac{1}{2} & \xi+\frac{1}{2} & -2\xi \end{bmatrix}}$$

$$J(\xi) = 2\xi(x_{ave} - x_3) + \frac{l_e}{2}$$

$\boxed{2\text{-NC}}$

Same integration with GQ:



$\xi_1^* = \frac{-1}{\sqrt{3}} \approx -.577\ldots$ $\qquad \xi_2^* = \frac{1}{\sqrt{3}} = .577\ldots$

$\omega_1 = 1 \qquad \omega_2 = 1$

TABLE 5.6 *Sampling points and weights in Gauss-Legendre numerical integration (interval −1 to +1)*

n	r_i	α_i
1	0. (15 zeros)	2. (15 zeros)
2	±0.57735 02691 89626	1.00000 00000 00000
3	±0.77459 66692 41483	0.55555 55555 55556
	0.00000 00000 00000	0.88888 88888 88889
4	±0.86113 63115 94053	0.34785 48451 37454
	±0.33998 10435 84856	0.65214 51548 62546
5	±0.90617 98459 38664	0.23692 68850 56189
	±0.53846 93101 05683	0.47862 86704 99366
	0.00000 00000 00000	0.56888 88888 88889
6	±0.93246 95142 03152	0.17132 44923 79170
	±0.66120 93864 66265	0.36076 15730 48139
	±0.23861 91860 83197	0.46791 39345 72691

$$k^e = \int_{-1}^{1} \underline{I}_{3\times3}(\xi)\,d\xi = \overbrace{\cancel{(1 - -1)}}^{\text{already incorporated to } \omega\text{'s in GQ}} \left(\underset{1}{\omega_1}\,\underline{I}_{3\times3}(\underset{-\frac{1}{\sqrt{3}}}{\xi_1^*}) + \underset{1}{\omega_2}\,\underline{I}_{3\times3}(\underset{\frac{1}{\sqrt{3}}}{\xi_2^*})\right)$$

$$\boxed{\Rightarrow \quad k^e = I_{3\times3}\left(-\frac{1}{\sqrt{3}}\right) + I_{3\times3}\left(\frac{1}{\sqrt{3}}\right) \qquad I_{3\times3}(\xi) = \frac{EA(\xi)}{J(\xi)} \begin{bmatrix} \xi-\frac{1}{2} \\ \xi+\frac{1}{2} \\ -2\xi \end{bmatrix} \left[\xi-\frac{1}{2} \quad \xi+\frac{1}{2} \quad -2\xi\right] \qquad \simeq 2\xi(X_{ave}-X_3)+\frac{l_e}{2}}$$

$\boxed{2\text{-}GQ}$

$n_G = 2$

better

fewer quadrature pts (50% here)

$$\boxed{k^e = \frac{1}{3}\left(I_{3\times3}(-1) + 4 I_{3\times3}(0) + I_{3\times3}(1)\right) \qquad I_{3\times3}(\xi) = \frac{EA(\xi)}{J(\xi)} \begin{bmatrix} \xi-\frac{1}{2} \\ \xi+\frac{1}{2} \\ -2\xi \end{bmatrix} \left[\xi-\frac{1}{2} \quad \xi+\frac{1}{2} \quad -2\xi\right] \qquad \simeq 2\xi(X_{ave}-X_3)+\frac{l_e}{2}}$$

$n_{NC} = 3$

$\boxed{2\text{-}NC}$

Q1 : EA, J constant, would eqns 2 give us the exact

$$k^e = \frac{AE}{L}\begin{bmatrix} 7/3 & -8/3 & 1/3 \\ & 7/3 & -8/3 \\ \text{sym} & & 16/3 \end{bmatrix} ?$$

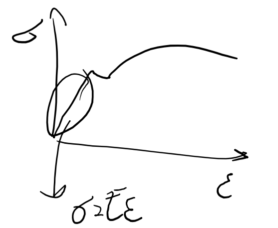
Yes: Because the integrand is a polynomial of

O: order $= 2$ & two schemes ($NC, n_{NC}=3$ or $G, n_G=2$)

both integrate polynomial order $=2$ exactly

Full Integration Order:
Number of quad points that integrate the element EXACTLY when:

- Element is homogeneous E A
- Constant section (1D and 2D)
- Not-skewed geometry (J = constant) J
- For nonlinear problem like solid mechanics we assume the material is linear in determining number of quad points.



$\hat{V}\ (V^2)$

poly order p $2 \times p$

integrand would be $3p$ order $= 15$

$$o_{full} = 2 \longrightarrow \begin{cases} n_{NC}^{full} = 3 \\ n_G^{full} = 2 \end{cases}$$

The integration will be exact is the actual integrand is a polynomial of o_full (in the problem shown A, E, J are all constant)

What if the element was:
- Skewed J not constant or
- Inhomogeneous E not constant or
- Non-prismatic A not constant

$$k^e = \int_{-1}^{1} I(\xi)\, d\xi$$

$$\underset{3\times3}{I(\xi)} = \frac{EA(\xi)}{J(\xi)} \begin{bmatrix} \xi - \frac{1}{2} \\ \xi + \frac{1}{2} \\ -2\xi \end{bmatrix} \begin{bmatrix} \xi - \frac{1}{2} & \xi + \frac{1}{2} & -2\xi \end{bmatrix}$$

$$= 2\xi(x_{arc} - x_3) + \frac{l_e}{2}$$

We almost always still use the full integration scheme!

no dof . unknowns

Reason:
Finite element or any numerical method has a huge source of error:
Discretization

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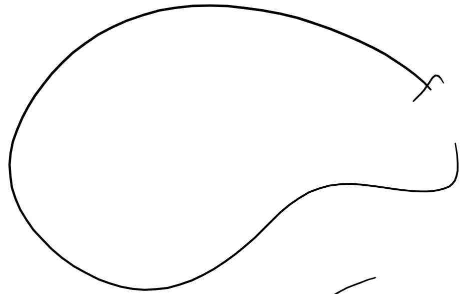
Discretization

∞ unknown



Disc. error

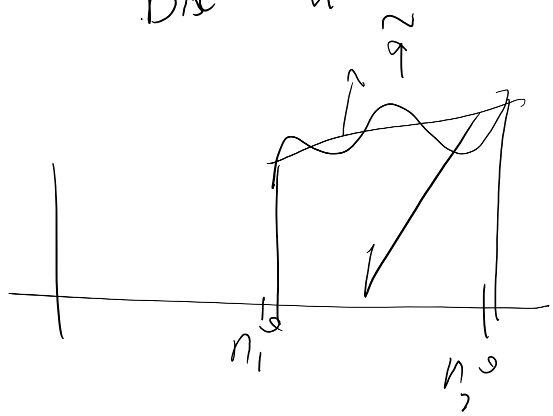
$$\text{error} \propto h^{p+1} \propto (n_f)^{-\frac{(p+1)}{d}}$$

$n_f \nearrow$ error $\searrow$

Any error we introduce whose error is similar or goes to zero faster than disc error is OK.

here $p=2$ Disc h^3

Example



Full order

TABLE 5.5 *Newton-Cotes numbers and error estimates*

Number of intervals n	C_0^n	C_1^n	C_2^n	C_3^n	C_4^n	C_5^n	C_6^n	Upper bound on error R_n as a function of the derivative of F
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3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$				$10^{-3}(b-a)^5F^{IV}(r)$
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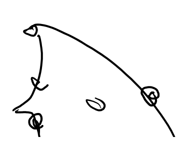
6	$\frac{41}{840}$	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$	$\frac{27}{840}$	$\frac{216}{840}$	$\frac{41}{840}$	$10^{-9}(b-a)^9F^{VIII}(r)$

h^5

Disc. h^3

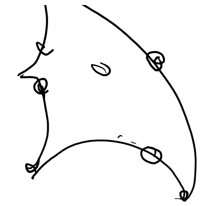
highly nonlinear



We recommend that *full numerical integration*[9] always be used for a displacement-based or mixed finite element formulation, where we define "full" numerical integration as

We recommend that *full numerical integration*[9] always be used for a displacement-based or mixed finite element formulation, where we define "full" numerical integration as the order that gives the exact matrices (i.e., the analytically integrated values) when the elements are geometrically undistorted. Table 5.9 lists this order for elements used in two-dimensional analyses.

nonlinear & skewed



use more than full order

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with $\underline{J \text{ non constant}}$ is there any any NC or G scheme that integrates this exactly

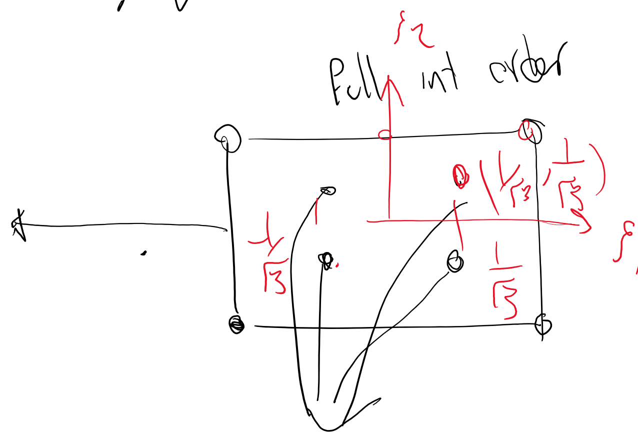
$$\underset{3\times3}{I(\xi)} = \frac{EA(\xi)}{J(\xi)} \begin{bmatrix} \xi-\frac{1}{2} \\ \xi+\frac{1}{2} \\ -2\xi \end{bmatrix} \begin{bmatrix} \xi-\frac{1}{2} & \xi+\frac{1}{2} & -2\xi \end{bmatrix}$$

$$J(\xi) = 2\xi(x_{ave}-x_3)+\frac{l_e}{2}$$

NO

Q: Can we even use a fewer number of quad points?

2D $p=1$ element



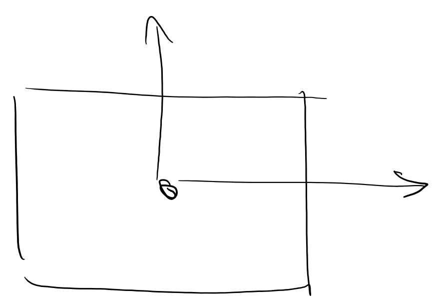
Gauss pts

Reduced integration

is an explicit scheme

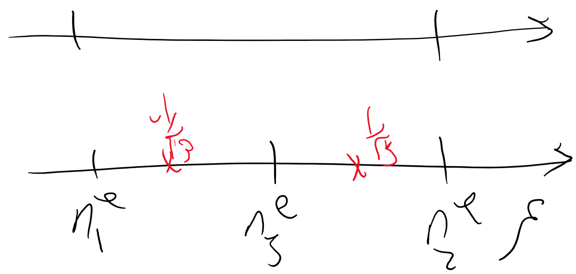
1 pt

$\frac{1}{4}$ comp time



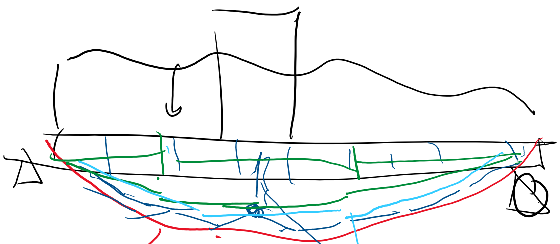
$$\boxed{k^e = 2\, I_{3\times3}\,(0)}$$

Cost Reason 1

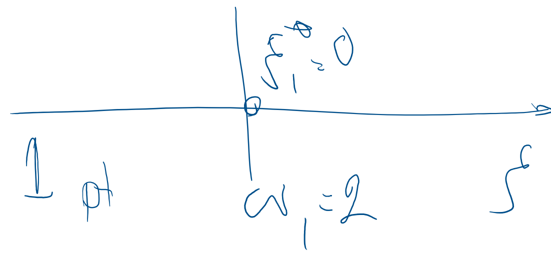


2nd motivation

FEM is stiff



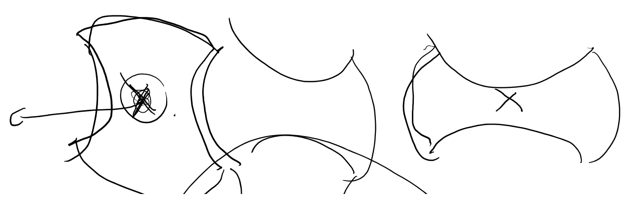
can we integrate it with 1 pt



with reduce order the response becomes more compliant

$\boxed{\text{more compliant Reason 2}}$

we create some nonphysical zero energy modes



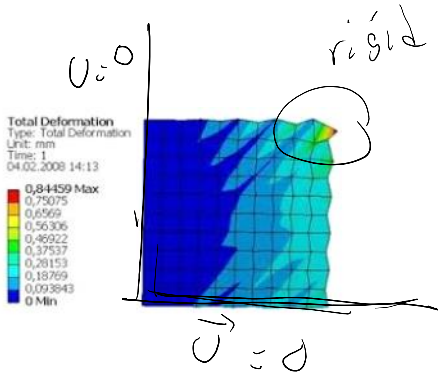
energy $= 0$

nonphysical zero mode

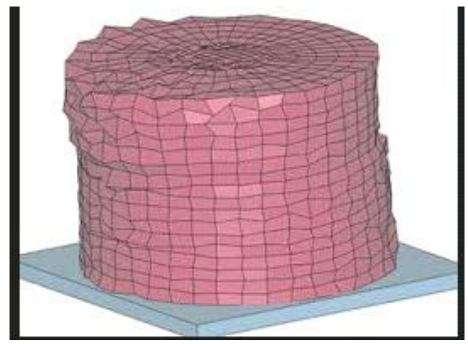
rigid

$k U = 0 U$

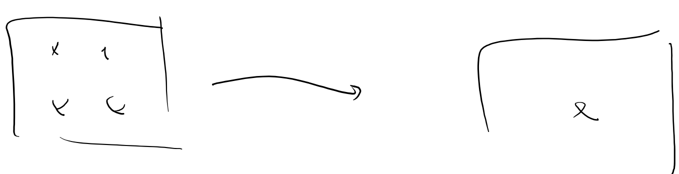
$U = 0$



$\vec{U} = 0$



In general, we don't want to do reduced order integration (anything lower than full integration order)
However, there are advanced techniques that afford going to reduced order integration.



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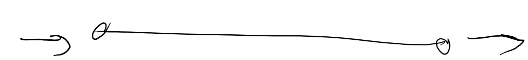
if we do reduced order integration, how can we know it's not too much & we don't introduce nonphysical zero modes

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \underbrace{\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{k^e} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



P=1 element

eigenvalues of k^e

eigenvalues of K

$$K^e U = \lambda U \iff \det(K - \lambda I) = 0$$

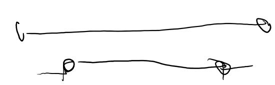
U: eigen vector; λ: eigenvalue

$$\det \begin{bmatrix} k-\lambda & -k \\ -k & k-\lambda \end{bmatrix} = 0 \qquad (k-\lambda)^2 - k^2 = 0 \qquad \left(k = \frac{AE}{L}\right)$$

$$\Rightarrow \quad k - \lambda = \pm k \quad \rightarrow \quad \begin{cases} \lambda = 0 \\ \lambda = 2k \end{cases}$$

$\lambda = 2k$

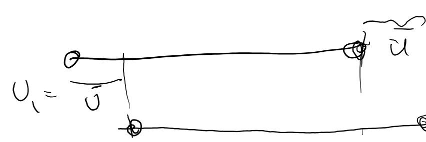
$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = 2k \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \quad \sim \quad U_2 = -U_1$$



$\lambda = 0$

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = 0 \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \qquad \Delta u = U_2 - U_1 = \bar{u} - \bar{u} = 0$$

$$\begin{bmatrix} k\,(U_1 - U_2) \\ k\,(-U_1 + U_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow U_1 = U_2$$



Rigid body motion

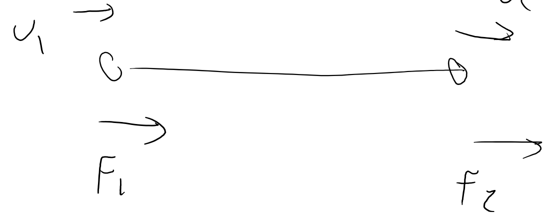
$$\varepsilon = \frac{\Delta U}{L_e} = 0$$

Zero energy mode

① Zero energy mode

Rigid motion $u(x) = \bar{u}$

$$\left(K^e_{2\times 2}\right)$$



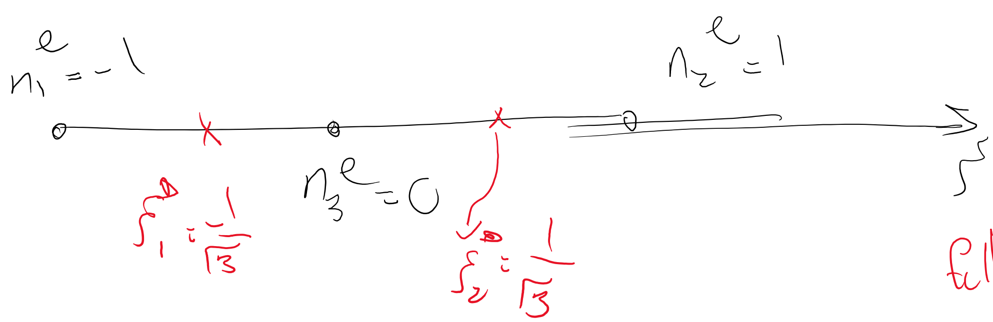
$$\text{rank}(K) = \underbrace{\dim(K)}_{\substack{\text{size of} \\ \text{matrix} \\ =2}} - \underbrace{\dim(\text{null}(K))}_{\#\text{ of zero eigenvalues}}$$

```
>> K = [1 -1; -1 1]

K =

     1    -1
    -1     1

>> rank(K)

ans =

     1
```

physically for a bar problem we must have 1 zero mode (rigid mode)

$\vec{u_1}$ ———— $\vec{u_2}$

$$\text{rank} = \underset{K_{2\times 2}}{2} - \underset{\text{1 rigid mode}}{1} = 1$$



full int order

$K_{3\times 3}$

$$\text{rank}(K) = \dim(K) - \#\text{ zero modes} = 3 - 1 = 2$$

Full int order gives rank = 2 :)

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \bar{u} \\ \bar{u} \\ \bar{u} \end{bmatrix}$$ is a zero mode rigid mode

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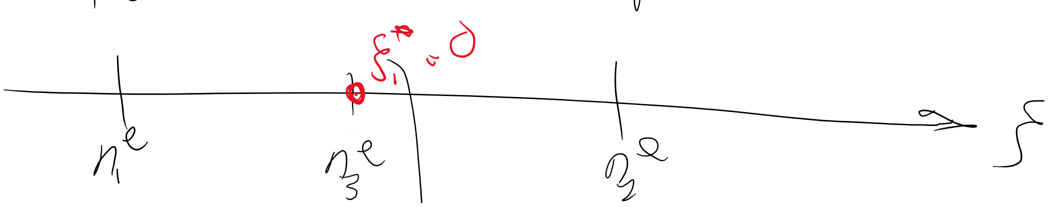
Reduced order integration



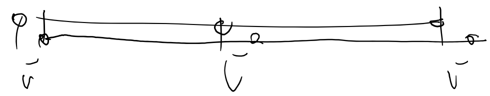
η_1^e $\quad$ η_3^e $\quad$ $\eta_2^e \longrightarrow \xi$

$$k^e = 2\, I_{3\times3}(0)$$

if we calculate the rank of this k we would get $\underline{\underline{1}}$ instead of $\underline{\underline{2}}$

Reason we have 2 (instead of 1) zero modes

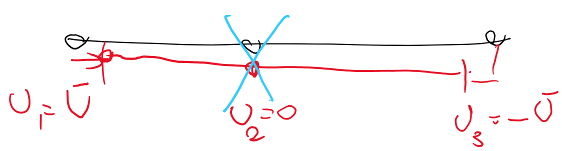
1)



$$U = \bar{u}\begin{bmatrix}1\\1\\1\end{bmatrix} \quad \text{rigid mode}$$

$$F = kU = \begin{bmatrix}0\\0\\0\end{bmatrix} \checkmark$$

2)



$$U = \bar{u}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$F = kU = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

nonphysical

Any coord tran

$$\det J \underline{\geq} 0$$

on a zero measure set

on a zero measure set

in 1D det $J = J > 0$

maybe at 1 pt