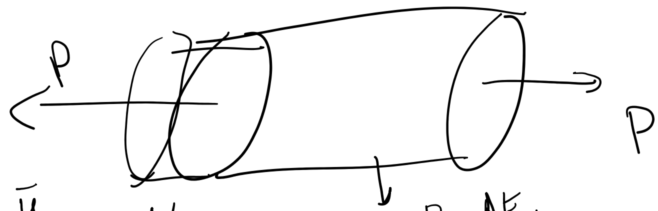
2022/09/08

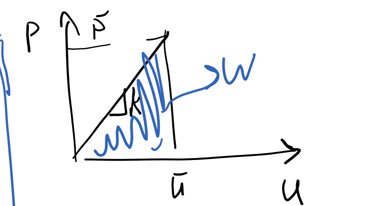
Thursday, September 8, 2022 2:32 PM

Calculating G for P-delta (u) systems

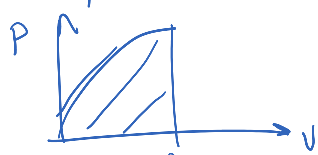


$$W = \int_0^{\bar{P}} P\,du = \int_0^{\bar{u}} k\,u\,du = \frac{1}{2} k\bar{u}^2 = \frac{1}{2}\underbrace{(k\bar{u})}_{\bar{P}}\bar{u}$$

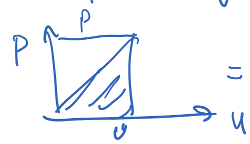
$P = \underbrace{\frac{AE}{L}}_{k} u$ ← change of displacement



For a P-u system W is area under P, u curve



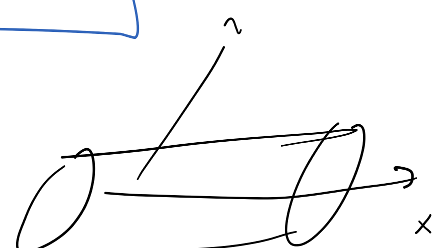
& if linear



$= \frac{1}{2} Pu$

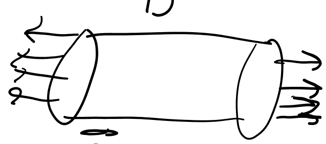
How about internal elastic energy

$$U_e = \int_D e(\varepsilon)\,dV$$



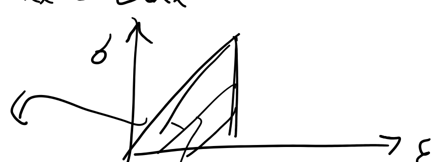
only $\varepsilon_{xx} \neq 0$ $\quad \nu = 0$

$\sigma_{xx} = E\varepsilon_{xx}$



$\sigma = \frac{\bar{P}}{A}$ Constant everywhere

linear elastic



$e(\varepsilon) = \frac{1}{2}\sigma\varepsilon = \frac{E}{2}\varepsilon^2$

$\varepsilon = \frac{\sigma}{E} = \frac{\bar{P}}{AE}$ " "

$$e(\varepsilon) = \frac{\sigma\varepsilon}{2} = \frac{1}{2}\left(\frac{\bar{P}}{A}\right)^2 \frac{1}{E}$$

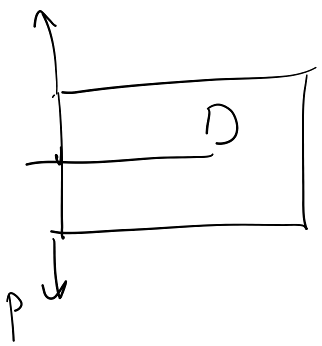
$$U_e = \int_{\text{Volume}} e(\varepsilon)\,dV = \underbrace{(\text{Volume})}_{AL}\left(\frac{1}{2}\frac{\bar{P}^2}{A^2 E}\right) = \frac{1}{2}\frac{L}{AE}\bar{P}^2$$

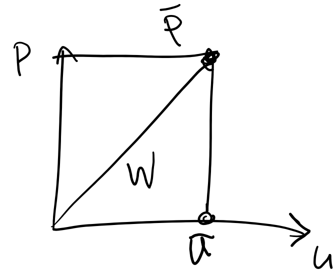
const

$$W = \frac{1}{2}\bar{P}\bar{u} = \frac{1}{2}\bar{P}\left(\frac{\bar{P}}{k}\right) = \frac{1}{2}\left(\frac{1}{k}\right)\bar{P}^2 = \frac{1}{2}\frac{L}{AE}\bar{P}^2$$

$\bar{P} = k\bar{u}$ $\qquad k = \frac{AE}{L}$

For more complex elastic loading we'd have W = Ue

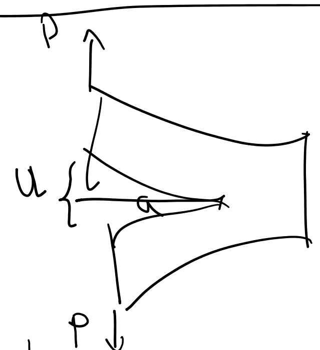




$$W = \frac{1}{2}\bar{P}\bar{u}$$

$$U_e = \int_D e(\varepsilon(x))\,dV = W$$

Usefulness of this



$$U_e = \frac{Pu}{2} = W_{o1}$$

no need to calculate $U_e = \int_D e(\varepsilon(x))\,dV$

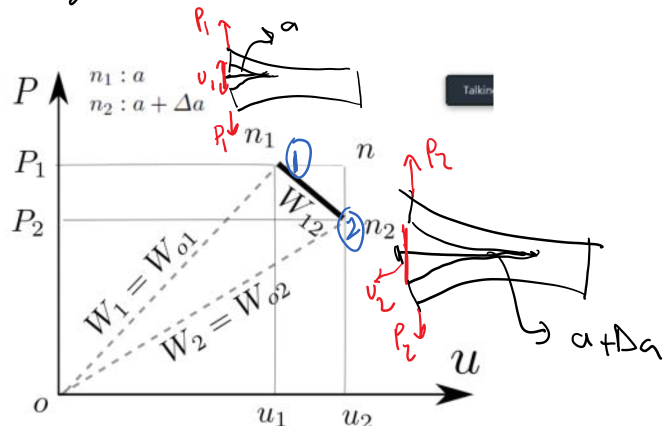
Given: A point load - displacement system with a crack and two data points:

- Load P_1, displacement u_1, & crack length a_1
- Load P_2, displacement u_2, & crack length $a_2 = a_1 + \Delta a$ (small Δa)

Goal: Compute G

Notation:

- W_{12}: External work from n_1 to n_2
- W_{o1} External work that would have happened through elastic (or almost elastic) deformation with fixed crack length from 0 to n_1.
- W_{o2} Similar to W_{o1}



$$G = -\frac{\Delta \Pi}{\Delta A}$$

$\Delta\Pi$ → change of Π

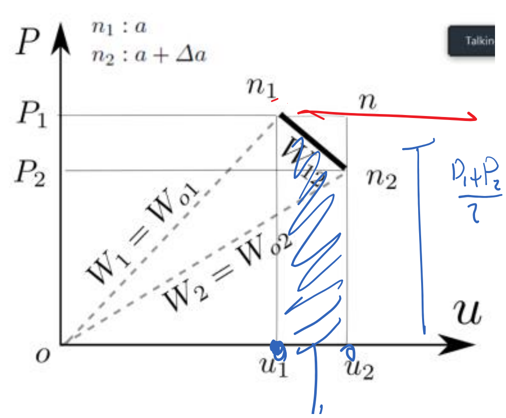
ΔA → change of area of the crack

$$\Pi = U_e - W$$

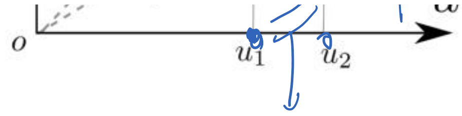
$$\Delta\Pi_{1\to 2} = \Pi_2 - \Pi_1$$

$$= (U_e)_2 - (U_e)_1 - W_{1\to 2}$$

$$W_{1\to 2} = \int_{u_1}^{u_2} P\,du \approx (u_2 - u_1)\left(\frac{P_1 + P_2}{2}\right)$$

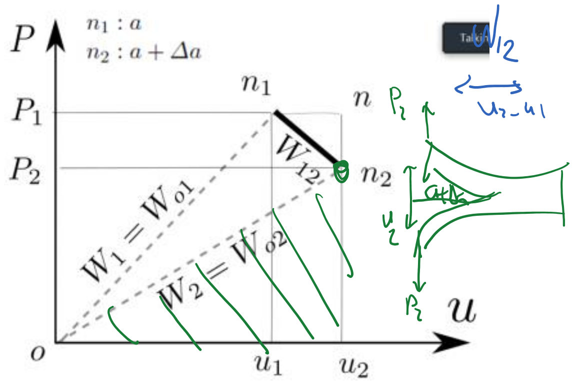


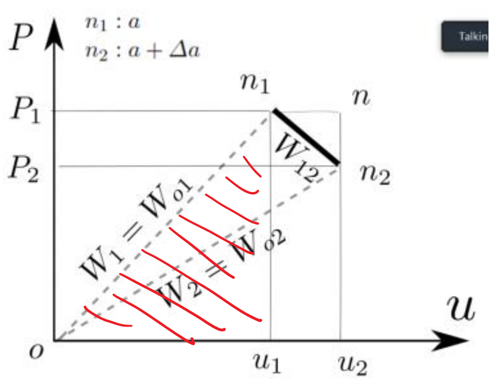
$$W_{1\to 2} = \int_{u_1}^{u_2} P\,du \approx (u_2 - u_1)\left(\frac{P_1+P_2}{2}\right)$$



$$(U_e)_2 = \frac{1}{2} P_2 u_2$$

$$(U_{e_1}) = \frac{1}{2} P_1 u_1$$





We'll get to these equations shortly, but for now, let's show that for fixed-grip and dead load cases, again G = shaded area / B / delta a

Fixed grip

$$\Delta\Pi = U_{e_2} - U_{e_1} - W_{12}$$

$$U_{e_2} = \frac{P_2 u}{2} \qquad U_{e_1} = \frac{1}{2} P_1 u$$

$$W_{12} = 0$$

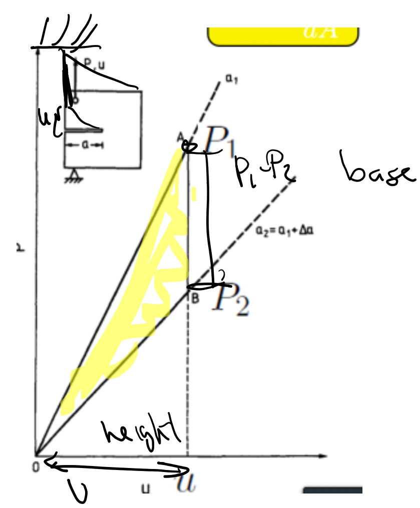
$$\Delta\Pi = \frac{(P_2 - P_1)u}{2}$$

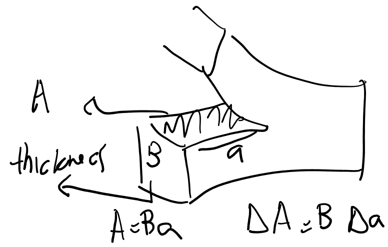
$$G = -\frac{\Delta\Pi}{\Delta A} = \frac{(P_1 - P_2)u}{2\,\Delta A}$$

$$G = \left(\frac{(P_1 - P_2)u}{2}\right)\frac{1}{B\,\Delta a}$$

$$\boxed{G = \frac{\text{shaded area}}{B\,\Delta a}}$$

fixed grip





Deadload

Dead loads

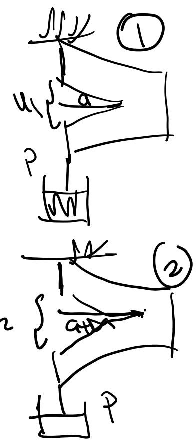
Deadload

$$\Delta\Pi = U_{e_2} - U_{e_1} - W_{12}$$

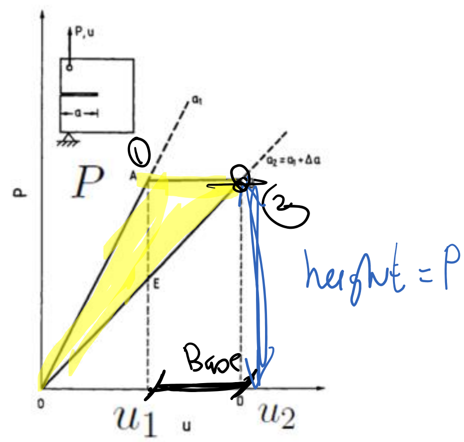
$$U_{e_2} = \frac{1}{2} P u_2$$

$$U_{e_1} = \frac{1}{2} P u_1$$

$$W_{12} = \int_{u_1}^{u_2} P\,du = P(u_2 - u_1)$$



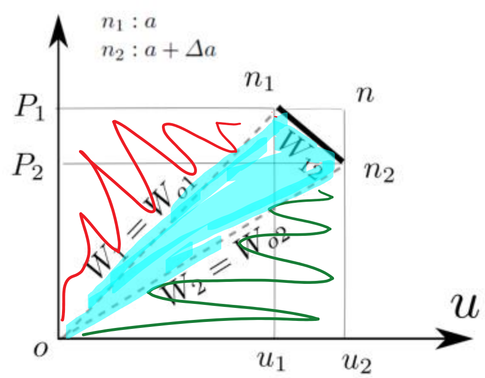
Dead loads



$$\Delta\Pi = \frac{1}{2} P u_2 - \frac{1}{2} P u_1 - P(u_2 - u_1) = -\left(\frac{1}{2} P(u_2 - u_1)\right) = -\text{shaded area}$$

$$G = -\frac{\Delta\Pi}{B\Delta a} \longrightarrow \boxed{G = \frac{\text{shaded area}}{B\Delta a}}$$

General case:

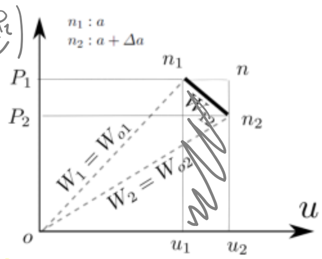


$$\Delta\Pi = U_{e_2} - U_{e_1} - W_{12}$$

$$U_{e_2} \rightarrow W_{o2} = \frac{1}{2} P_2 u_2$$

$$U_{e_1} \rightarrow W_{o1} = \frac{1}{2} P_1 u_1$$

$$W_{12} = (u_2 - u_1)\left(\frac{P_1 + P_2}{2}\right)$$



$$= \frac{1}{2} P_2 u_2 - \frac{1}{2} P_1 u_1 - (u_2 - u_1)\left(\frac{P_1 + P_2}{2}\right)$$

$$= \frac{1}{2} P_2 u_2 - \frac{1}{2} P_1 u_1 - \frac{1}{2} P_1 u_2 - \frac{1}{2} P_2 u_2 + \frac{1}{2} u_1 P_1 + \frac{1}{2} u_1 P_2$$

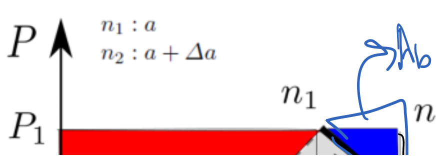
$$\boxed{\Delta\Pi = -\frac{1}{2}(P_1 u_2 - P_2 u_1) \qquad G = -\frac{\Delta\Pi}{\Delta A} = \frac{P_1 u_2 - P_2 u_1}{2 B \Delta a}}$$

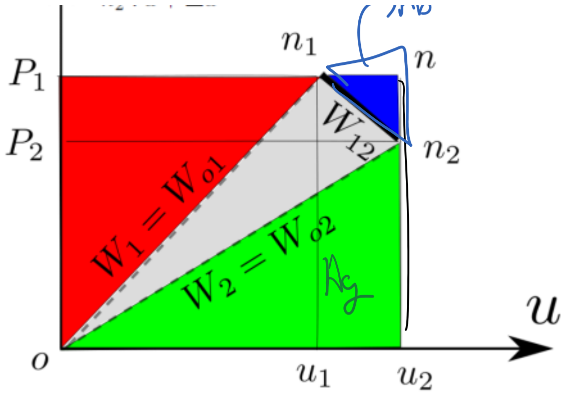
HW 1 problem

to prove $G = \frac{P^2}{2B} \frac{dC}{da}$ $\quad \boxed{u = CP}$ compliance

need to show $\frac{1}{2}(P_1 u_2 - P_2 u_1)$ is the shaded area

shaded area = total rectangle area $- A_g - A_r$
$- A_b$





$$= P_1 u_2 - \underbrace{\tfrac{1}{2} P_2 u_2}_{U_{o_2}} - \underbrace{\tfrac{1}{2} P_1 u_1}_{U_{e_1}}$$

$$- \tfrac{1}{2}(P_1 - P_2)(u_2 - u_1)$$

$$= \tfrac{1}{2}(P_1 u_2 - P_2 u_1)$$

show at hw

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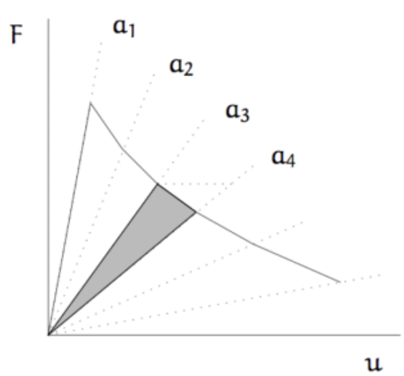
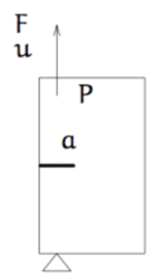
shaded area method → experimentally measure $G = \frac{\text{shaded area}}{B \, \Delta a}$

→ $G = R$ obtain R

stable c. propagation





$$\boxed{G(a_3) = \frac{1}{B} \frac{\text{shaded area}}{a_4 - a_3}}$$

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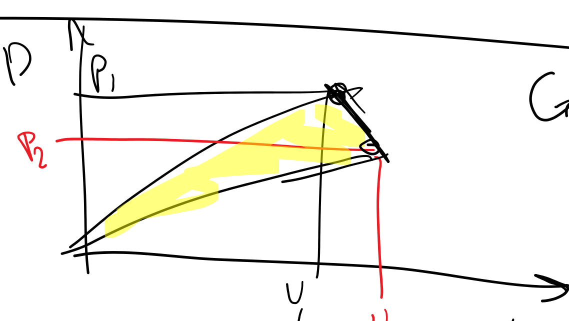
HW



$$G = \lim_{\Delta a \to 0} \frac{\text{shaded area}}{B \Delta a} = \lim_{\Delta a \to 0} \frac{\frac{1}{2}(P_1 u_2 - P_2 u_1)}{B \Delta a} \quad ☆$$

let $\Delta a \to 0$

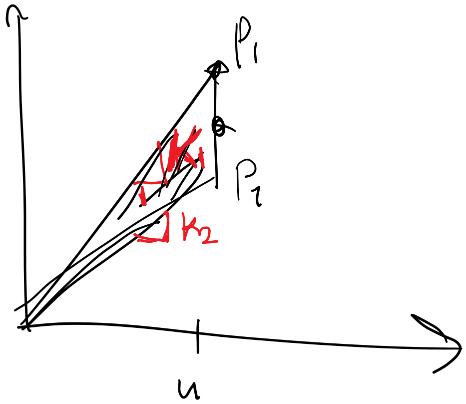
$$\boxed{G = \frac{P^2}{2B} \frac{dC}{da}} \rightarrow \text{compliance}$$

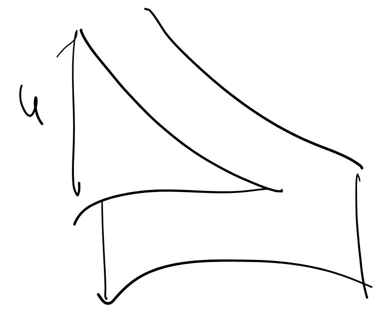
u_2 u $G = \frac{1}{2B}\frac{du}{da}$ → compliance → any loading

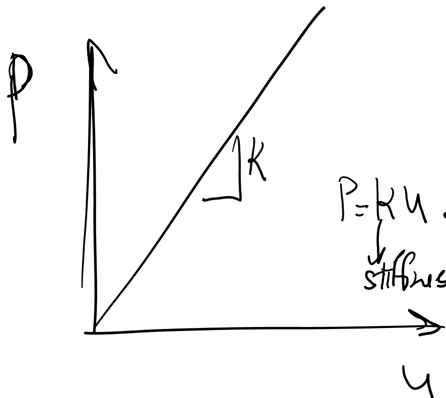
I'll show this for fixed grip & dead load

Fixed grip

$$\boxed{G = \frac{\frac{1}{2}(P_1 - P_2)\,u}{B\Delta a}}$$







$P = ku \Longleftrightarrow u = CP$

stiffness (k) — compliance (C)

$$\begin{cases} P_1 = \dfrac{u}{C_1} \\ P_2 = \dfrac{u}{C_2} \end{cases} \rightarrow G = \lim_{\Delta a \to 0} \frac{\frac{1}{2}\left(\frac{u}{C_1} - \frac{u}{C_2}\right) u}{B\Delta a} = \frac{-u^2}{2B}\lim_{\Delta a \to 0}\left(\frac{1/C_2 - 1/C_1}{\Delta a}\right)$$

$$= \frac{-u^2}{2B}\frac{d\left(\frac{1}{C}\right)}{da} = \frac{-u^2}{2B}\,\frac{-1}{C^2}\,\frac{dC}{da} = \underbrace{\left(\frac{u}{C}\right)}_{P}^2 \frac{1}{2B}\frac{dC}{da}$$

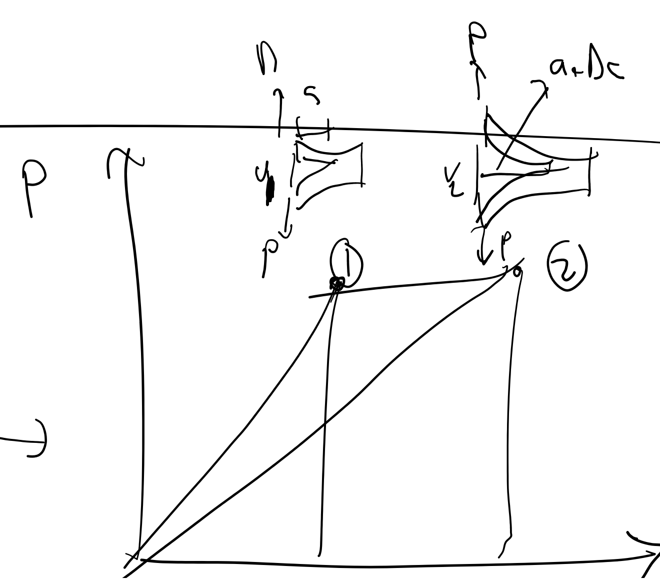
$$\boxed{G = \frac{P^2}{2B}\frac{dC}{da}}$$ Fixed grip

Dead load

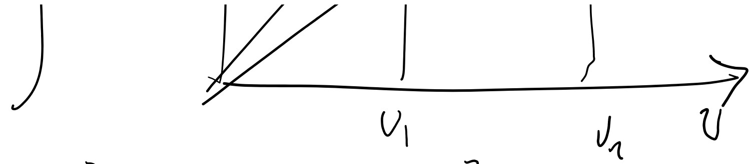
$$G = \frac{\text{shaded area}}{B\Delta a} = \frac{\frac{1}{2}P(u_2 - u_1)}{B\Delta a}$$

$u_2 = C_2 P$

$u_1 = C_1 P$

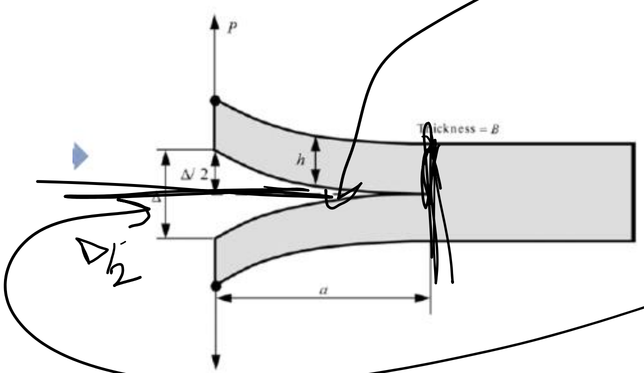


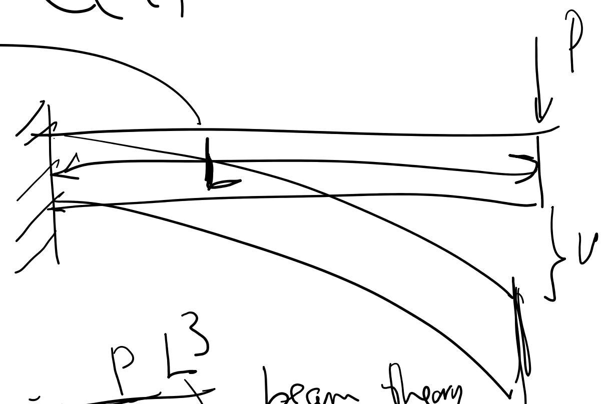
$$u_2 = C_2 P$$
$$u_1 = C_1 P$$



$$G = \lim_{\Delta a \to 0} \frac{\frac{1}{2} P (C_2 P - C_1 P)}{B \Delta a} = \frac{\frac{1}{2} P^2}{B} \lim \frac{C_2 - C_1}{\Delta a} = \boxed{\frac{P^2}{2B} \frac{dC}{da}} \text{ dead load}$$

Example of how to calculate $C(a)$





$$u = \frac{P L^3}{3EI} \text{ beam theory}$$

$$L \to a$$

$$\frac{\Delta}{2} = \frac{P a^3}{3EI} \qquad \boxed{C(a) = \frac{\Delta}{P} = \frac{2a^3}{3EI}}$$

$$G = \frac{P^2}{2B} \frac{dC}{da} = \frac{P^2}{2B} \left(\frac{2}{3} \times 3 \frac{a^2}{EI} \right)$$

$$\boxed{G = \frac{P^2 a^2}{BEI}}$$