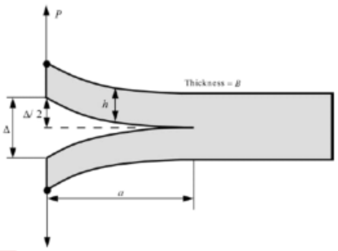
2022/09/13

Tuesday, September 13, 2022 2:24 PM

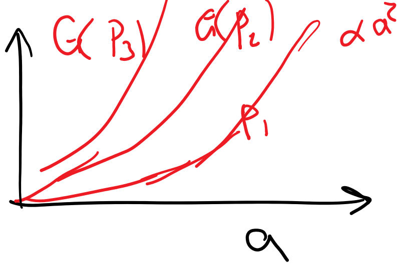
From last time:

$$\frac{\Delta}{2}=\frac{Pa^3}{3EI}$$

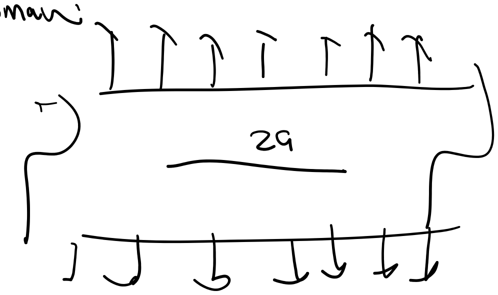
$$C=\frac{\Delta}{P}=\frac{2a^3}{3EI}$$



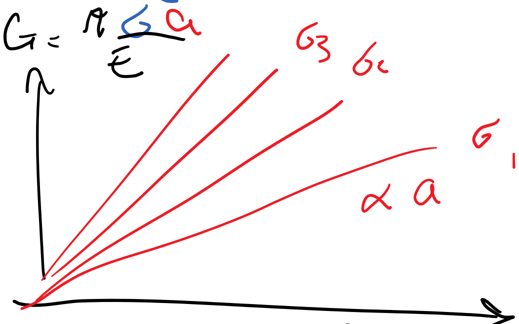

$$\boxed{G=\frac{P^2}{2B}\frac{dC}{da}=\frac{P^2a^2}{BEI}}$$



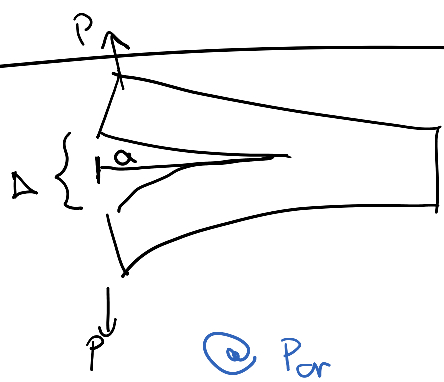

Compare with mid crack example in an infinite domain:

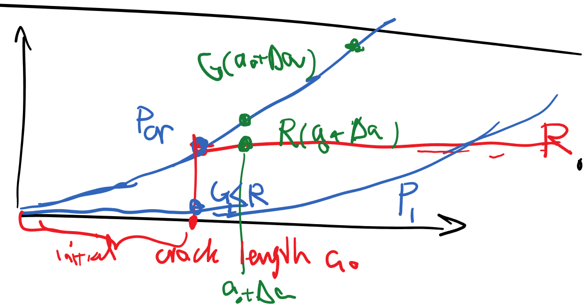



$$G=\frac{\pi\sigma^2 a}{E}$$




slower G increase versus a




@ $P_{cr}$ $\quad G=R$

$$\frac{P_{cr}^2a_0^2}{EI}=R_0 \rightarrow P_{cr}=\frac{\sqrt{EI\,R_0}}{a_0}$$

continued crack propagation

$G(a_0+\Delta a)\geq R(a_0+\Delta a)$

we already have

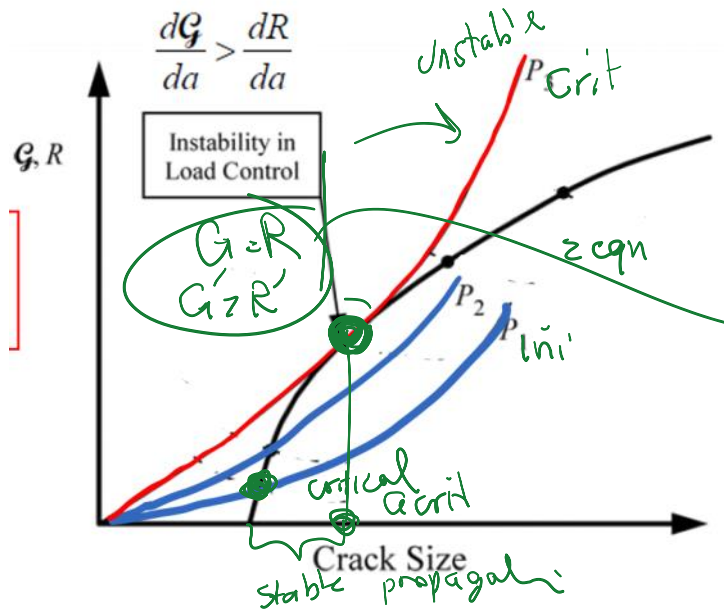
$G(a_0)=R(a_0)$

$$\rightarrow \lim_{\Delta a\to 0}\frac{G(a_0+\Delta a)-G(a_0)}{\Delta a}\geq\frac{R(a_0+\Delta a)-R(a_0)}{\Delta a}$$

$$\boxed{G'(a_0)\geq R'(a_0)}$$

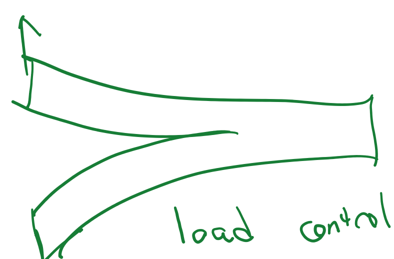
only needed @ critical point

From critical pt we need this condition for continued prop.

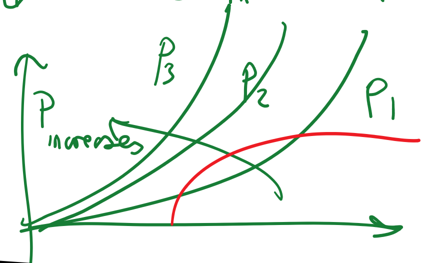



2 unknowns $a_{crit}$ & $P_{crit}$

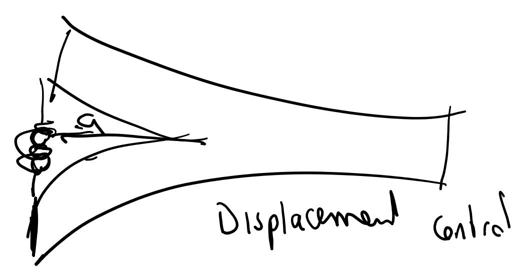
## Displacement control




we increase P




we control $\Delta$




$\Delta$ is controled

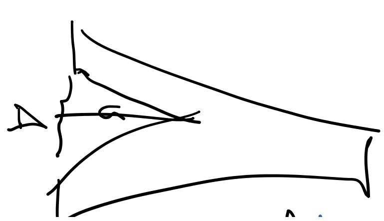
$$G = \frac{P^2 a^2}{EI}$$

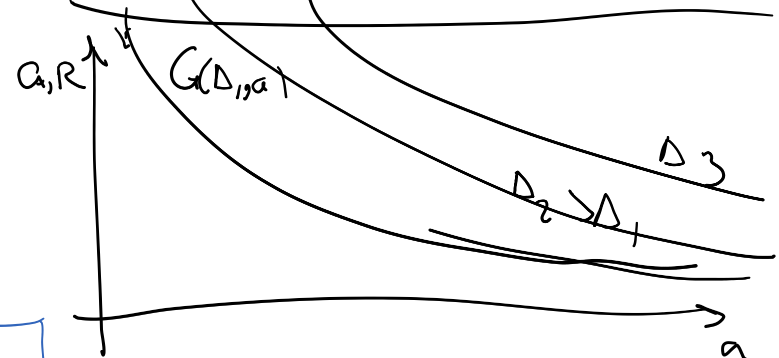
$$P \;\&\; \Delta: \quad P = \frac{3}{2}\frac{EI\Delta}{a^3}$$

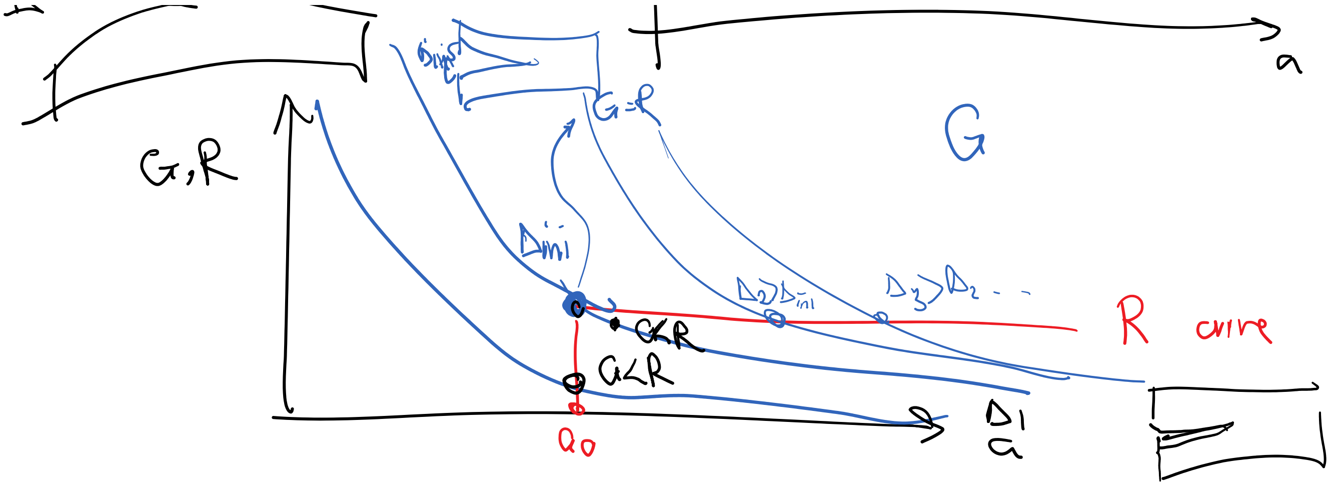
$$G = \left(\frac{3}{2}\frac{EI\Delta}{a^3}\right)^2 \frac{a^2}{EI}$$

for displacement control

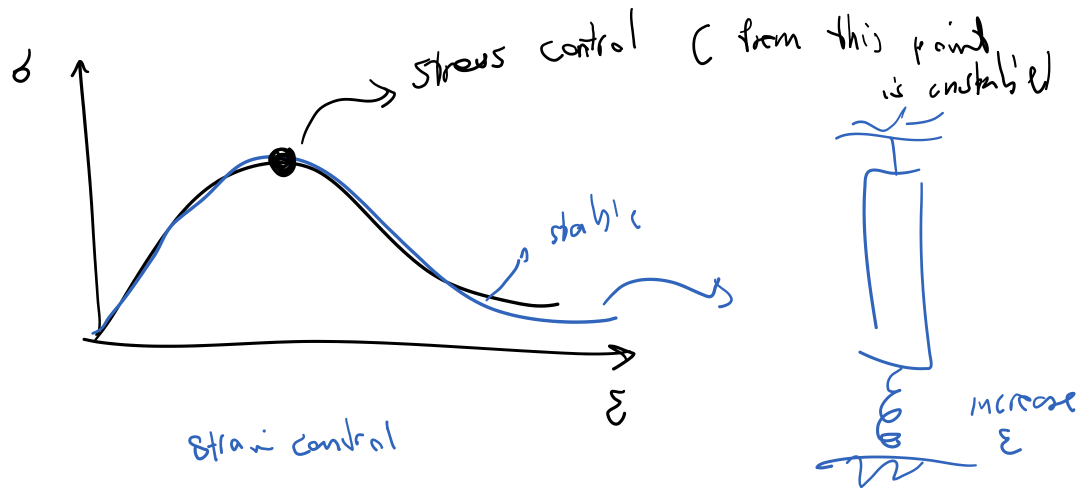
$$G = \frac{9}{4}\frac{EI\Delta^2}{a^4}$$



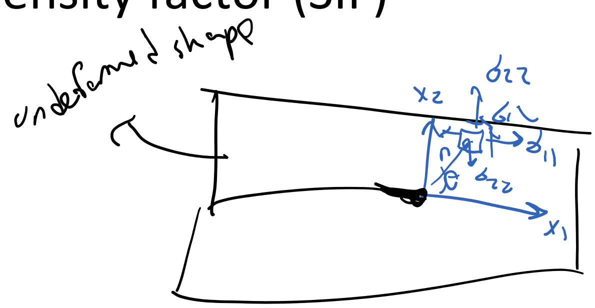



We need to increase the applied displacement Δ for the crack to continue propagation -> stable crack growth




# Stress intensity factor (SIF)

1) Local asymptotic stress solution




$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} \sum_{22}^{I}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} \sum_{22}^{II}(\theta) + \frac{K_{III}}{\sqrt{2\pi r}} \sum_{22}^{III}(\theta)$$

$$\sigma_{ij} \propto \frac{1}{\sqrt{r}}$$ for a sharp crack

$\sigma_{ij} \propto \dfrac{1}{\sqrt{r}}$ for a sharp crack

local / stress
use of it

Global / energy

$$G = \frac{1}{E'}\left(K_I^2 + K_{II}^2\right) + \frac{K_{III}^2}{\mu}$$ !

compute E.R.R (G) from K's

Typical approach for solving elastodynamic problems:

acceleration

1) Equation of motion $\nabla \cdot \sigma + \rho b = \rho \ddot{u}$

2) Constitutive eqn $\sigma = C \varepsilon$ generalization of 1D $\sigma = E\varepsilon$

3) Compatibility eqn $\varepsilon = \frac{1}{2}\left(\nabla u + \nabla u^t\right)$

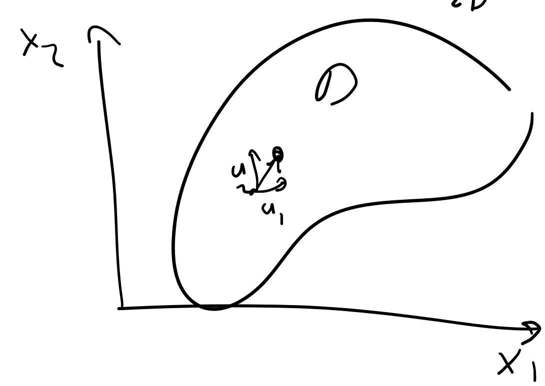
$$\nabla \cdot C \nabla u + \rho b = \rho \ddot{u}$$

2nd order in time

2nd order in space

and it's a vector equation $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

2D



# Displacement approach

EOM $\sigma_{ij,j} = 0$

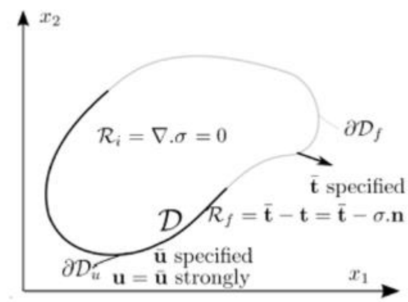
Const eqn

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk,j}\right)$$

$$\frac{E}{1+\nu}\left(\varepsilon_{ij,j} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk,j}\right) = 0$$

compatibility

$$\varepsilon_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$




$$\underbrace{(\lambda+\mu)\nabla\nabla.\mathbf{u} + \mu\nabla.\nabla\mathbf{u}}_{\nabla.C\nabla u} = \mathbf{0} \quad \text{or} \quad (\lambda+\mu)u_{j,ji} + \mu u_{i,jj} = 0$$

$\mu, \lambda$ are used for isotropic solid

$\nabla . C\nabla u + \rho b = \rho \ddot{u}$ , $\rho b = 0$ (no body force), $\rho\ddot{u} = 0$ static

BC's

PDE + BC for $\mathbf{u} \Rightarrow \varepsilon \Rightarrow \sigma$ ↳ $\sigma = C\varepsilon$

(1) $\varepsilon = \frac{\nabla u + \nabla u^T}{2}$

$\rho b$ — $b$: force per mass, Ex. gravity $b = -g e_z$

body force per volume

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Reverse direction

$\sigma \longrightarrow \varepsilon = C^{-1}\sigma \longrightarrow u$ = some integral of $\varepsilon$

$\sigma$ ↓ from a Diff equation

Isotropic 2D solid

This is done using Airy stress function :

2D

$\psi(x_1, x_2)$

$$\sigma_{ij} = -\psi_{,ij} + \delta_{ij}\psi_{,kk}$$

Airy stress function

indicial notation
$a_i a_i := \sum_{i=1}^{3} a_i a_i$
we drop this

$$\sigma_{11} = -\psi_{,11} + \delta_{11}(\underbrace{\psi_{,11} + \psi_{,22}}_{\psi_{,kk}})$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$$= \cancel{-\psi_{,11}} + (\cancel{\psi_{,11}} + \psi_{,22}) = \psi_{,22}$$

Similarly $\sigma_{22} = \psi_{,11}$

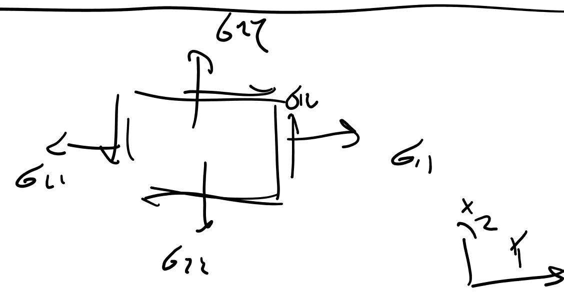
Similarly $\sigma_{22} = \Psi_{,11}$

$$\sigma_{12} = -\Psi_{,12} + \delta_{12}^{\nearrow 0} (\Psi_{,11} + \Psi_{,22}) = -\Psi_{,12}$$

Summary $\Psi(x_1, x_2)$ is given $\rightsquigarrow$

step 1

$$\boxed{\begin{aligned} \sigma_{11} &= \Psi_{,22} \\ \sigma_{22} &= \Psi_{,11} \\ \sigma_{12} &= -\Psi_{,12} \end{aligned}}$$




step 2 $\quad \varepsilon = C^{-1} \sigma$

step 3 $\quad$ derive $u$ from $\varepsilon$ $\quad \left( \varepsilon = \frac{\nabla u + \nabla u^T}{2} \right)$

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Example

How about $\quad \Psi(x_1, x_2) = x_1^4$


step 1

$$\sigma_{11} = \Psi_{,22} = 0$$

$$\sigma_{22} = \Psi_{,11} = 12 x_1^2$$

$$\sigma_{12} = -\Psi_{,12} = 0$$

assume $\nu = 0$

step 2 Calculate strains $\quad (\nu = 0)$

$$\left[ \begin{aligned} \varepsilon_{11} &= \frac{\sigma_{11}}{E} = 0 \\ \varepsilon_{22} &= \frac{\sigma_{22}}{E} = \frac{12 x_1^2}{E} \\ \varepsilon_{12} &= \frac{\sigma_{12}}{G} = 0 \end{aligned} \right.$$

Step 3 $\quad$ calculate displacements

eq1) $\varepsilon_{11} = u_{1,1} = 0 \rightsquigarrow \quad u_1(x_1, x_2) = f(x_2)$

eq2) $\varepsilon_{22} = u_{2,2} = \frac{12 x_1^2}{E} \rightsquigarrow \quad u_2(x_1, x_2) = \frac{12 x_1^2 x_2}{E} + g(x_1)$

eq2) $\varepsilon_{22} = u_{2,2} = \frac{12x_1^2}{E}$ $\rightarrow$ $u_2(x_1, x_2) = \frac{12x_1^2 x_2}{E} + g(x_1)$ $\int$

eq3) $\varepsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1}) = 0$

plug in eq3) $u_{1,2} + u_{2,1} = f'(x_2) + \frac{24x_1x_2}{E} + g'(x_1) = 0$

only a function of $x_2$

function of $x_1x_2$

only a function of $x_1$

Impossible

We cannot use any stress function

Problem

3 strains

$\varepsilon_{11} = u_{1,1}$

$\varepsilon_{22} = u_{2,2}$

$\varepsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1})$

2 displacements $u_1$, $u_2$

?

not possible in general

In 2D, the problem is that we have 3 equations and 2 unknowns (u1 and u2). We may not be able to solve u1, u2 in general.

Can we always obtain u from strain?
No

3D) 6 strains -> 3 displacements

3 compatibility conditions are needed

$$\boxed{\frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_j} + \frac{\partial^2 \varepsilon_{jj}}{\partial x_i \partial x_k} - \frac{\partial^2 \varepsilon_{jk}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{ij}}{\partial x_j \partial x_k} = 0.}$$

2D): There is only 1 equation

compatibility condition $\boxed{\varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} = 0}$ (*)

compatibility condition in 2D $\boxed{\epsilon_{11,22} + \epsilon_{22,11} - 2\epsilon_{12,12} = 0}$ (★)

I show use $u \to \epsilon \longrightarrow$ satisfies (★)

$$\left.\begin{array}{lll} \epsilon_{11} = u_{1,1} & \longrightarrow & \epsilon_{11,22} = u_{1,122} \\ \epsilon_{22} = u_{2,2} & \longrightarrow & \epsilon_{22,11} = u_{2,211} \\ \epsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1}) & \longrightarrow & 2\epsilon_{12,12} = u_{1,212} + u_{2,112} \end{array}\right] \to \epsilon_{11,22} + \epsilon_{22,11} - 2\epsilon_{12,12} = 0$$

:)

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Which Airy function approach

$$\psi \to \begin{pmatrix} \sigma_{11} = \psi_{,22} \\ \sigma_{22} = \psi_{,11} \\ \sigma_{12} = -\psi_{,12} \end{pmatrix} \xrightarrow{\text{calculate strain}}$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\{-\psi_{,ij} + (1-\nu)\delta_{ij}\underbrace{\psi_{,kk}}_{\psi_{,11}+\psi_{,22}}\}$$

$\longrightarrow$ check compatibility condition

$\epsilon_{11,22} + \epsilon_{22,11} - 2\epsilon_{12,12} = 0$

$$(\psi_{,11} + \psi_{,22})_{,11} + (\psi_{,11} + \psi_{,22})_{,22} = 0 \quad (★★)$$

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$$\Delta\psi = \nabla\cdot\nabla\psi = \psi_{,11} + \psi_{,22}$$

$$\Delta\psi = 0 \longrightarrow \text{Harmonic function}$$

$$\boxed{\Delta\Delta\psi = 0} \quad \text{biharmonic function}$$

stress approach

Displacement approach

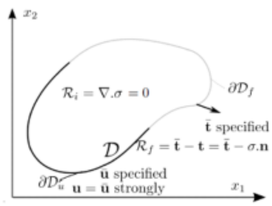
## Displacement approach

$$\sigma_{ij,j} = 0$$

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk,j}\right)$$

$$\frac{E}{1+\nu}\left(\varepsilon_{ij,j} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk,j}\right) = 0$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$




$(\lambda+\mu)\nabla\nabla.\mathbf{u} + \mu\nabla.\nabla\mathbf{u} = 0$ or $(\lambda+\mu)u_{j,ji} + \mu u_{i,jj} = 0$

BC's

2nd order :)
vector eqn :( $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

PDE + BC for $\mathbf{u} \Rightarrow \epsilon \Rightarrow \sigma$

Stress approach

$\Delta\Delta\psi = 0$ $\quad (\psi_{,1111} + \psi_{,2222} + 2\psi_{,1122} = 0$

$\sigma \to \varepsilon \to u$

$\sigma_{11} = \psi_{,22}$

4th order :(

scalar eqn :)

$\psi$

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Complex functions provide many harmonic solutions

$$f(z) = z^2 = (x+iy)^2 = (x^2-y^2) + i(2xy)$$

harmonic $\qquad$ harmonic

$U(x,y) = x^2 - y^2 \qquad U_{,xx} + U_{,yy} = 2-2=0$

$V(x,y) = 2xy$
$V_{,xx} + V_{,yy} = 0$

$z^3, z^4, \sin z$