2022/09/15

Thursday, September 15, 2022 2:21 PM

Recall, we need to solve $\Delta\Delta\psi = 0$

Fact: Real & Imaginary part of any complex funct— are harmonic

$z = x + iy$

$f(z) = U(x,y) + i\,V(x,y)$

complex; real part of $f(z)$; imag part

$\Delta U = 0, \quad \Delta V = 0 \rightarrow 1$

$f_{,x} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = f' \frac{\partial(x+iy)}{\partial x} = f'(z)$

$f_{,xx} = f''(z)$

$f_{,y} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial f}{\partial z}\frac{\partial(x+iy)}{\partial y} = \frac{\partial f}{\partial z}(i) = i f'(z)$

$f_{,yy} = \frac{\partial f'(z)}{\partial z}(i) = (i)^2 f''(z) = -f''(z)$

$\rightarrow f_{,xx} + f_{,yy} = 0$

$f_{,xx} + f_{,yy} = \underbrace{(U_{,xx} + U_{,yy})}_{\text{real}} + i\underbrace{(V_{,xx} + V_{,yy})}_{\text{imaginary}} = 0 + i0$

$$U_{,xx} + U_{,yy} = 0$$
$$V_{,xx} + V_{,yy} = 0$$

$U = \mathrm{Re}\, f(z)$

$V = \mathrm{Im}\, f(z)$

Complex functions are also used to seek biharmonic solutions

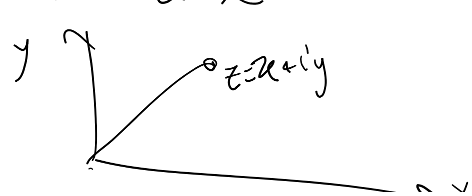
Stress function approach

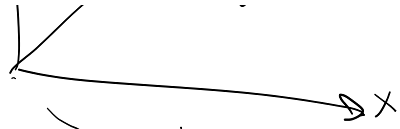
- Any biharmonic solution can be expressed by Kolonov-Muskhelishvili complex potentials, ϕ, χ:

$\Delta\Delta\psi = 0 \implies \Psi(x_1, x_2) = \mathrm{Re}\left[\bar{z}\phi + \chi\right]$

there are two complex functions ϕ & χ

$\bar{z}$ conjugate of z





Choose reasonable ϕ & $\chi \rightarrow \psi \rightarrow \sigma \rightarrow \varepsilon \rightarrow u$ $\quad \bar{z} = x - iy$

We will derive mode I and II solutions using this approach today.

$$\psi = \mathrm{Re}(\bar{z}\phi + \chi)$$

- Stresses are obtained differentiation,

$$\sigma_{11} = \Psi_{,22} = \mathrm{Re}\left[\phi' - \frac{1}{2}\bar{z}\phi'' - \frac{1}{2}\chi''\right]$$

$$\sigma_{22} = \Psi_{,11} = \mathrm{Re}\left[\phi' + \frac{1}{2}\bar{z}\phi'' + \frac{1}{2}\chi''\right]$$

$$\sigma_{12} = -\Psi_{,12} = \frac{1}{2}\mathrm{Re}\left[\bar{z}\phi'' + \chi''\right]$$

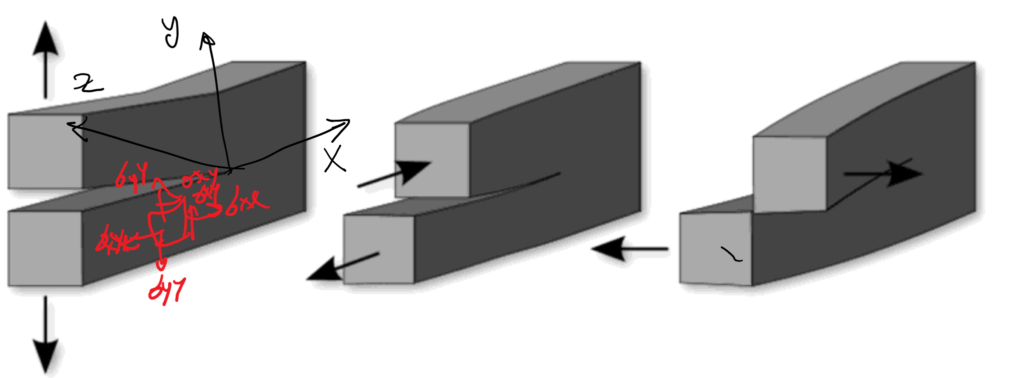
- Displacements are obtained by integration of strains:

$$u_1 = \mathrm{Re}\left[\kappa\phi - \bar{z}\phi' - \chi'\right]$$

$$u_2 = \mathrm{Im}\left[\kappa\phi + \bar{z}\phi' + \chi'\right]$$

$$\kappa = \begin{cases} 3 - 4\nu & \text{plane strain} \\ \frac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$$

3 Modes of fracture



Mode I: Opening | **Mode II:** In-plane shear | **Mode III:** Out-of-plane shear

In-plane modes

in-plane modes
$u_x(x,y) \neq 0 \quad u_y(x,y) \neq 0$

$\varepsilon_{zz} = 0$
$u_z = 0$ plane strain
$\sigma_{zz} = 0$ plane stress

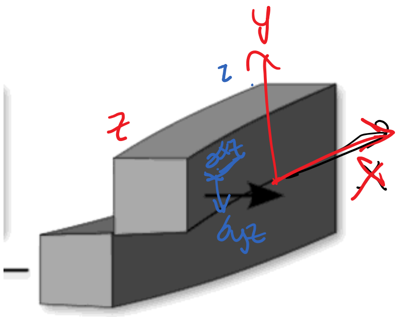
Vector problems
$u = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$

in-plane
$\sigma_{xx} \neq 0 \quad \sigma_{yy} \neq 0 \quad \sigma_{xy} \neq 0$
$\varepsilon_{xx} \neq 0 \quad \varepsilon_{yy} \neq 0 \quad \sigma_{xy} \neq 0$

out of plane
$\sigma_{xz} = 0 \quad \sigma_{yz} = 0$
σ_{zz} & ε_{zz}

mode III

mode III



Mode III:
Out-of-plane shear

$u_x = 0$, $u_y = 0$

$u_z(x,y) \neq 0$

in plane strain & stress is zero

$\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0$

$\epsilon_{xx} = \epsilon_{yy} = \epsilon_{xy} = 0$

out of plane

$\sigma_{zz} = \epsilon_{zz} = 0$

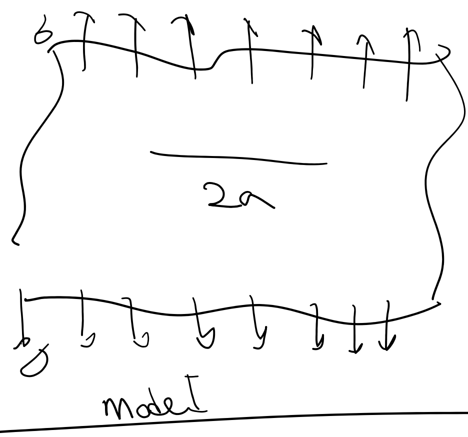
$\sigma_{xz} \neq 0$, $\sigma_{yz} \neq 0$

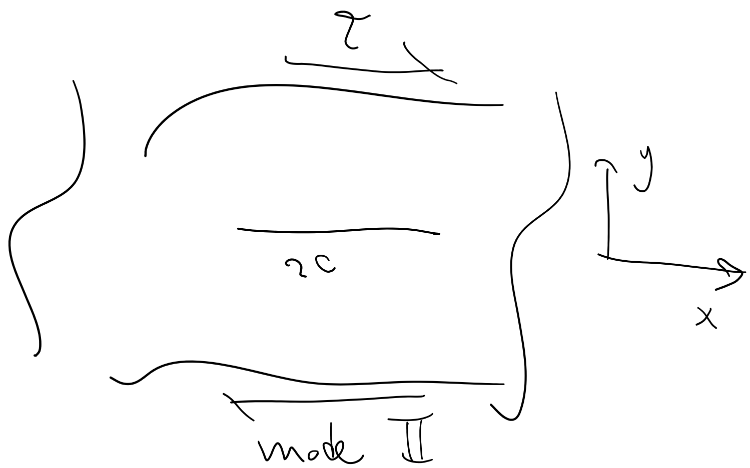
ϵ_{xz}, $\epsilon_{yz} \neq 0$

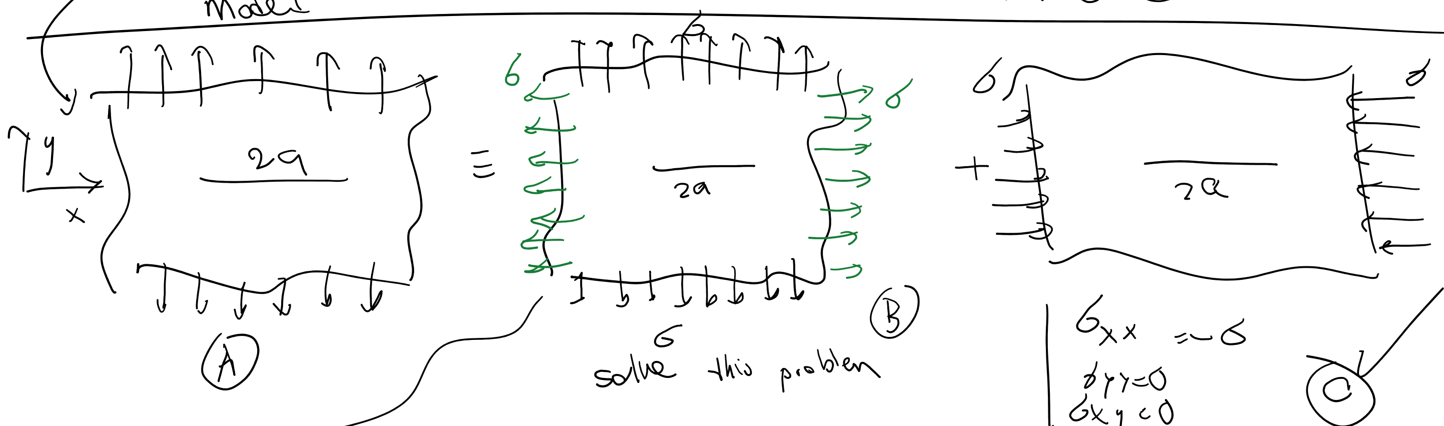
Scalar problem

easier to solve

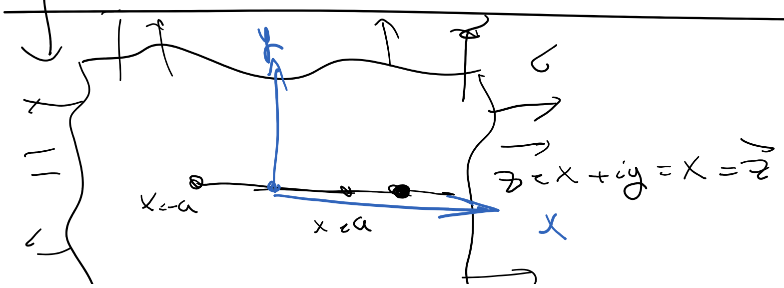
Getting the solution for Modes I and II: Mid-crack problem

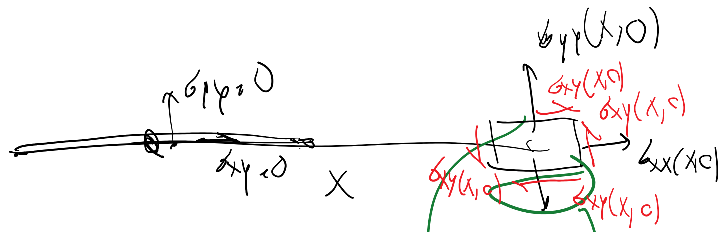


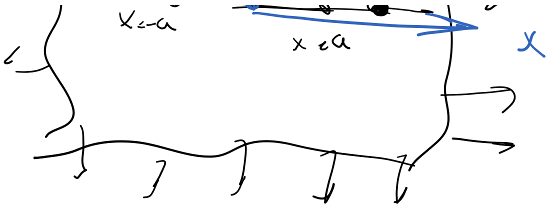


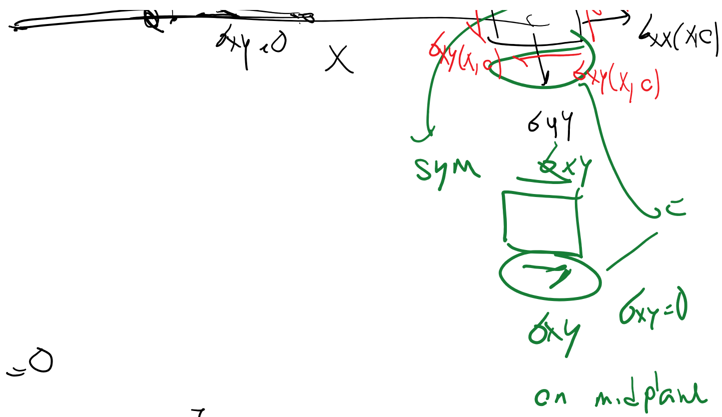


So, by solving problem (B) we subtract σ from σ_{xx} to have the solution for (A)









$$\sigma_{xy} = \frac{1}{2}\,\mathrm{Re}\left(\bar{z}\,\phi'' + \chi''\right)$$

$$\sigma_{xy}(x,0) = 0 \longrightarrow \frac{1}{2}\,\mathrm{Re}\left(\bar{z}\,\phi'' + \chi''\right) = 0$$

on x axis

$z\phi'' + \chi''$ what if

$$z\phi'' + \chi'' = 0 \longrightarrow \boxed{\chi'' = -z\phi''}$$ integrate this

$$\chi' = -z\phi' + \phi$$ integrate one more time

$$① \quad \boxed{\chi = -z\phi + 2\tilde{\phi}}, \quad \tilde{\phi}' = \phi \quad (\tilde{\phi} = \text{anti-derivative (integral) of } \phi$$

if χ & ϕ satisfy relation ① this guarantees that on x axis

$\sigma_{xy} = 0 \quad (\sigma_{xy}(x,0) = 0)$

eqn ① for χ

$$\psi = \mathrm{Re}(\bar{z}\phi + \chi) = \mathrm{Re}(\bar{z}\phi - z\phi + 2\tilde{\phi})$$

$$= \mathrm{Re}\big(\underbrace{(x - iy)}_{\bar{z}}\phi - (x+iy)\phi + 2\tilde{\phi}\big) = \mathrm{Re}(-2iy\,\phi + 2\tilde{\phi})$$

$$\boxed{\psi = 2\left[y\,\mathrm{Im}\,\phi + \mathrm{Re}\,\tilde{\phi}\right]} \quad ②$$

$z = x + iy$

Imaginary part, Real part

$\mathrm{Re}\, z = x$

$\mathrm{Im}\, z = y$

Use notation $\boxed{2\phi = Z(z)}$ ③ $\quad (\tilde{Z}' = Z)$

Z is a complex function

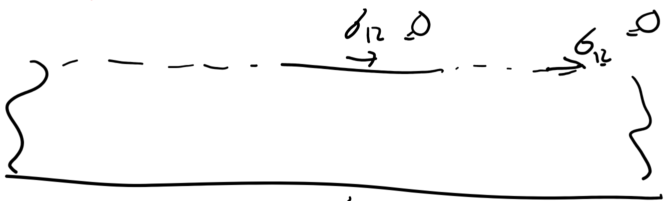
② & ③ $\rightarrow \quad \psi = y\,\mathrm{Im}\,\tilde{Z} + \mathrm{Re}\,\tilde{\tilde{Z}}$

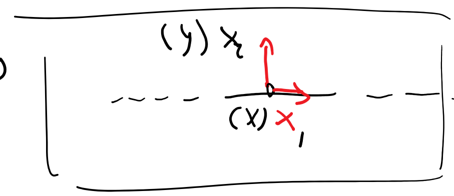
$$\psi'' = \mathrm{Re}\,Z \quad , \quad \mathrm{Im}\,Z'$$

$$\rightarrow \quad \begin{aligned} \sigma_{11} &= \psi_{,22} = \mathrm{Re}\,Z - y\,\mathrm{Im}\,Z' \\ \sigma_{22} &= \psi_{,11} = \mathrm{Re}\,Z + y\,\mathrm{Im}\,Z' \\ \sigma_{12} &= -\psi_{,12} = -y\,\mathrm{Re}\,Z' \end{aligned} \quad (4)$$

Now, all is left is just to find a suitable function Z that satisfies all boundary conditions.

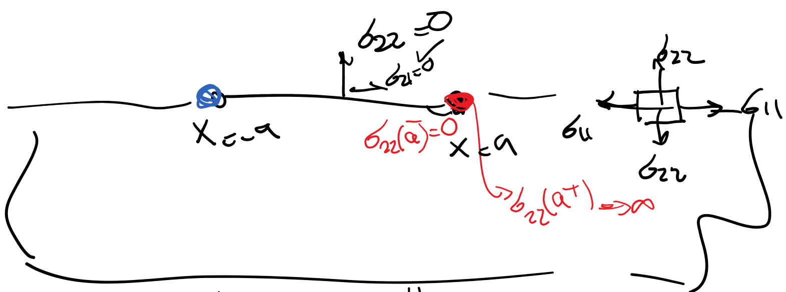
Let's focus on the crack surface $(y=0)$





(4) $y=0 \rightarrow \sigma_{12} = -y\,\mathrm{Re}\,Z' = 0$

Let's find a good candidate for Z



$\sigma_{22} \rightarrow \sigma$ on the crack surface $y=0$

$x \rightarrow \infty \rightarrow z = x + iy = x$

let's just work with $z \rightarrow x$ for our arguments for the form of Z

$$\sigma_{22}(x,0) = \mathrm{Re}\,Z - y\,\mathrm{Im}\,Z' = \mathrm{Re}\,Z \quad (y \rightarrow 0)$$ on the crack surface

proposal $Z = \dfrac{\sigma}{1 - a/x}$ on the crack surface use $z \rightarrow x$

$$Z(x) = \frac{\sigma}{1 - a/x}$$

$x \rightarrow a^{+}$ $\sigma_{22} = \mathrm{Re}\,Z(x) \rightarrow \infty$ ✓

$x \rightarrow \infty$ $\sigma_{22} = \mathrm{Re}\,Z(x) = \sigma$ ✓

to have symmetry & right solution for $x<0$

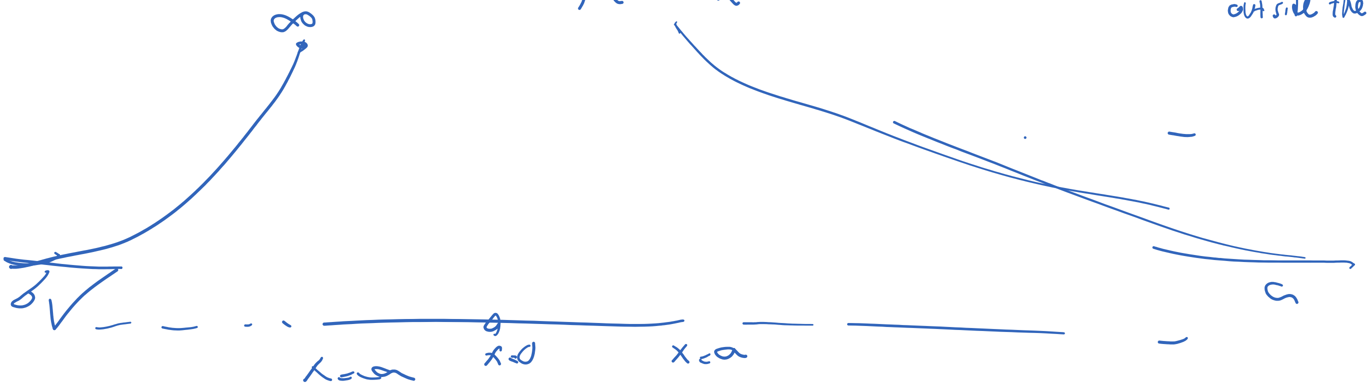
$$Z(x) = \frac{\sigma}{\quad\quad}$$

$x \rightarrow \pm\infty$ $\sigma_{22} \rightarrow \sigma$ ✓

$$Z(x) = \frac{\sigma}{\left(1-\frac{a}{x}\right)\left(1+\frac{a}{x}\right)}$$

$x \to \pm\infty \quad \sigma_{22} \to \sigma$ ✓

$x \to a$ (just outside the crack) $\sigma_{22} \to \infty$



$$\sigma_{22} = \mathrm{Re}\, Z = \mathrm{Re}\, \frac{\sigma}{\left(1-\frac{a}{x}\right)\left(1+\frac{a}{x}\right)}$$

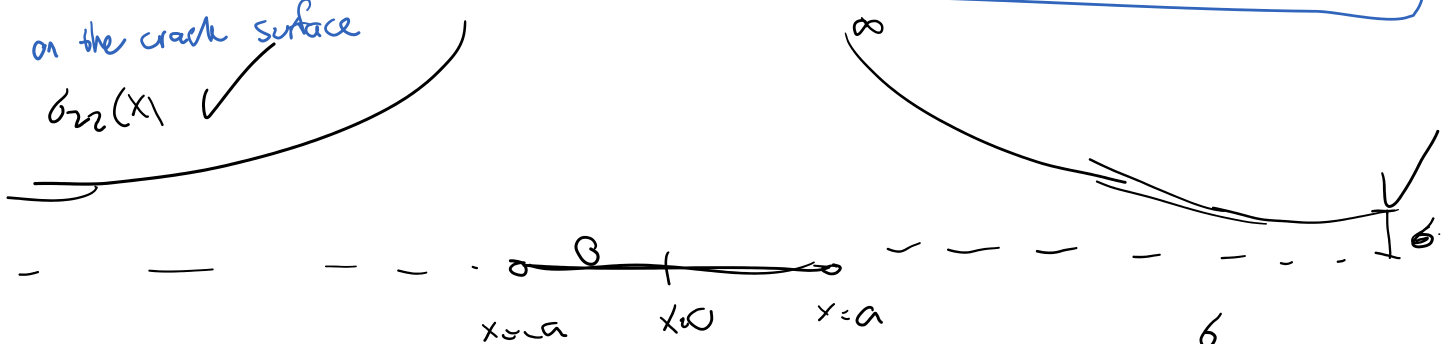
to get $\sigma_{22} = 0$, $-a < x < a$ take the square root

Propose $x \to z$

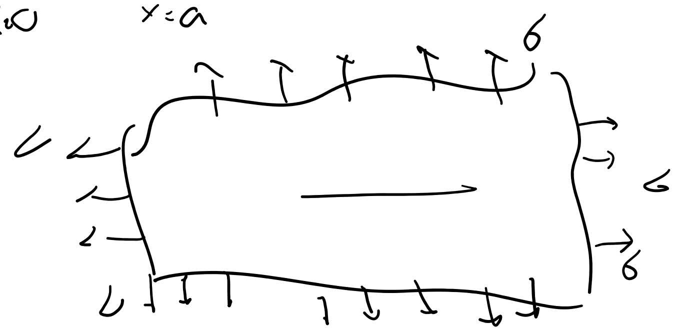
$$Z(z) = \frac{\sigma}{\sqrt{\left(1-\frac{a}{z}\right)\left(1+\frac{a}{z}\right)}} = \frac{\sigma}{\sqrt{1-\frac{a^2}{z^2}}}$$

on the crack surface

$\sigma_{22}(x)$ ✓



for field



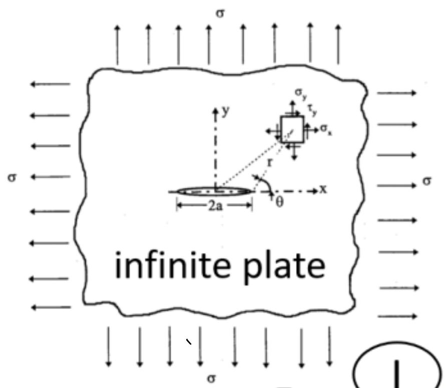
$$(x,y) \to \infty : \sigma_{xx} = \sigma_{yy} = \sigma, \tau_{xy} = 0$$

$$|x| < a, y = 0 : \sigma_{yy} = \tau_{xy} = 0$$

boundary conditions

$$Z(z) = \frac{\sigma z}{\sqrt{z^2 - a^2}}$$

$$\sigma_{xx} = \mathrm{Re}Z - y\mathrm{Im}Z'$$
$$\sigma_{yy} = \mathrm{Re}Z + y\mathrm{Im}Z'$$
$$\tau_{xy} = -y\mathrm{Re}Z'$$



I

II

$y = 0,\ |x| < a$

$$Z(z) = \frac{\sigma x}{\sqrt{x^2 - a^2}}$$ is imaginary

$$Z(z) = \frac{\sigma}{\sqrt{1 - (a/z)^2}}$$

$$Z'(z) = -\frac{\sigma a^2}{(z^2 - a^2)^{3/2}} \to 0$$

This is a global solution, valid at every point, but unfortunately only applies to this particular problem

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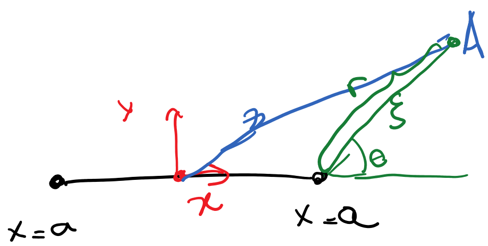
Now, let's look at the neighborhood of the crack tip



$$\xi = re^{i\theta} = r\cos\theta + r\sin\theta\, i$$

$z = a + \xi$ Find asymptotic solution around the crack tip

$\frac{|\xi|}{a} \ll 1$ we're very close to the crack

$r \ll a$

$$Z(z) = \frac{\sigma}{\sqrt{1-\left(\frac{a}{z}\right)^2}} = \frac{\sigma z}{\sqrt{z^2 - a^2}} \quad , z = a + \xi$$

$$= \frac{\sigma a(1+\xi/a)}{\sqrt{\xi^2 + 2a\xi}} = \frac{\sigma a(1+\xi/a)}{\sqrt{2a\xi(1+\xi/2a)}}$$

we're looking @ $\frac{\xi}{a} \ll 1$

$Z(z) = Z(\xi) \approx \sigma\sqrt{\pi a}$ ... around the crack tip

$$Z(z) = Z(\xi) \approx \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi\xi}}$$ around the crack tip

$\sigma\sqrt{\pi a}$ → stress intensity factor

$$K_I = \sigma\sqrt{\pi a}$$

$$Z(z) = Z(\xi) \approx \frac{K_I}{\sqrt{2\pi\xi}} \qquad \frac{|\xi|}{a} \ll 1, \quad K_I = \sigma\sqrt{\pi a}$$

Approximate equation for stress around the crack tip

$$\sigma_{22} = \mathrm{Re}\, Z - y\, \mathrm{Im}\, Z'$$

$$Z' = \frac{dZ}{dz} = \frac{dZ}{d\xi}\frac{d\xi}{dz} = \frac{dZ}{d\xi}\frac{d(z-a)}{dz} = \frac{dZ}{d\xi} = \frac{K_I}{\sqrt{2\pi}}\left(-\frac{1}{2}\right)\xi^{-3/2}$$