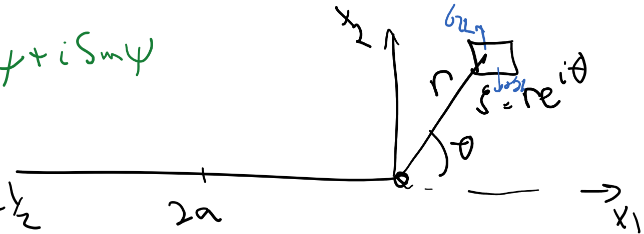
2020/09/20

Tuesday, September 20, 2022 2:31 PM

From last time

Euler's identity $e^{i\psi} = \cos\psi + i\sin\psi$



the asymptotic solution

$$Z(\zeta) = \frac{K_I}{\sqrt{2\pi\zeta}} = \frac{K_I}{\sqrt{2\pi}} \left(re^{i\theta}\right)^{-\frac{1}{2}}$$

$$\sigma_{22} = \mathrm{Re}\,\underbrace{\left(\frac{K_I}{\sqrt{2\pi}} \frac{1}{\sqrt{r}} \left(\cos\frac{\theta}{2} + i\sin\left(-\frac{\theta}{2}\right)\right)\right)}_{Z}$$

$$\sigma_{22} = \mathrm{Re}\, Z - y\, \mathrm{Im}\, Z'$$

$$- \underbrace{(r\sin\theta)}_{y}\, \mathrm{Im}\left(\frac{K_I}{\sqrt{2\pi}}\left(-\frac{1}{2}\right)\frac{1}{r\sqrt{r}}\left(\cos\frac{3\theta}{2} + i\sin\left(-\frac{3\theta}{2}\right)\right)\right)$$

$$Z' = \frac{K_I}{\sqrt{2\pi}}\left(-\frac{1}{2}\right)\zeta^{-\frac{3}{2}}$$

$$y = r\sin\theta \qquad \left(re^{i\theta}\right)^{-\frac{3}{2}} = \frac{1}{\sqrt{r^3}}\left(\cos-\frac{3\theta}{2} + i\sin\left(-\frac{3\theta}{2}\right)\right)$$

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{22} = \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1 + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$

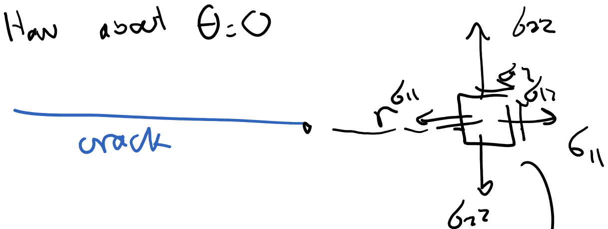
inverse square root

$\frac{1}{\sqrt{r}}$ singularity

$r \to 0 : \sigma_{ij} \to \infty$

121

How about $\theta = 0$

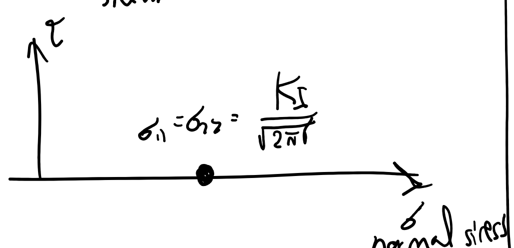


$\to$ Mohr Circle

$$\sigma_{11} = \sigma_{22} = \frac{K_I}{\sqrt{2\pi r}}$$

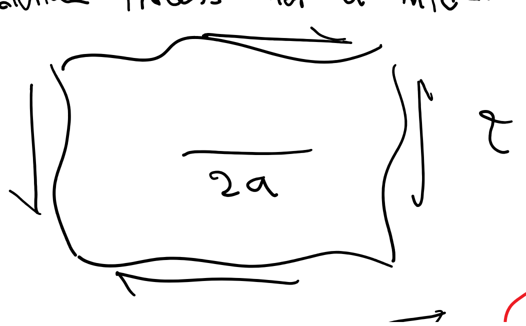
$$\sigma_{12} = 0$$

shear stress



Mode II

Exact solution Process for a mid-crack example



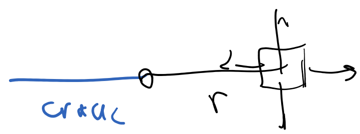
mode II: stress function $Z = \frac{-i\tau z}{\sqrt{a^2 - z^2}}$

mode I recall we had $Z = \frac{\sigma z}{\sqrt{a^2 - z^2}}$

we get mode II solution as

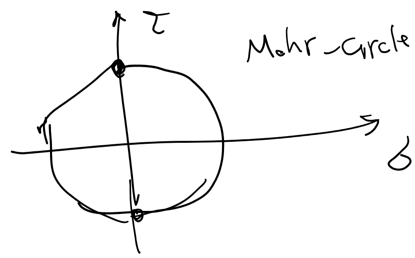
$$\boxed{\begin{aligned}\sigma_{xx} &= -\frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)\\ \sigma_{yy} &= \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\\ \tau_{xy} &= \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\end{aligned}}$$

$$\boxed{K_{II} = \tau\sqrt{\pi a}}$$



$\sigma_{xx} = \sigma_{yy} = 0$

$\sigma_{xy} = \frac{K_{II}}{\sqrt{2\pi r}}$



Mohr circle

Summary

Universal nature of the asymptotic stress field

Westergaards, Sneddon etc.

$$\sigma_{11} = \sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\underbrace{\left[1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]}_{f_{11}^{I}(\theta)}$$

$$\sigma_{22} = \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1 + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{12} = \tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \quad f_{12}^{I}(\theta)$$

(mode I)

$$f_{11}^{II}(\theta)$$

$$\boxed{\begin{aligned}\sigma_{xx} &= -\frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)\\ \sigma_{yy} &= \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\\ \tau_{xy} &= \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\end{aligned}}$$

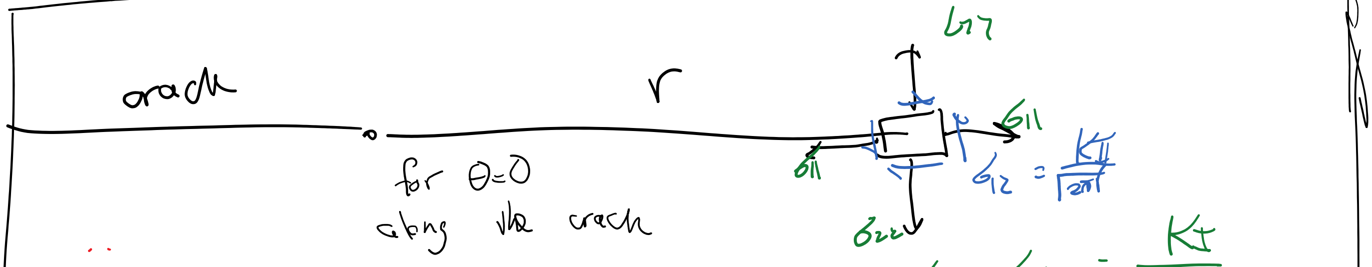
(mode II)

Irwin $\quad \sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta) + \text{H.O.T}$

$$= \frac{K_I}{\ } f^{(I)}(\theta) + K_{II} f^{(II)}(\theta) + K_{III} f_{ij}^{III}(\theta)$$

... with ... $\sqrt{2\pi r}\, f_{ij}(\theta)$ + H.O.T.

$$\sigma_{11} = \frac{K_I}{\sqrt{2\pi r}} f_{11}^{(I)}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f_{11}^{(II)}(\theta) + \frac{K_{III}}{\sqrt{2\pi r}} f_{11}^{III}(\theta)$$



$$\sigma_1 = \sigma_{22} = \frac{K_I}{\sqrt{2\pi r}}$$

→ How to calculate K's & define them

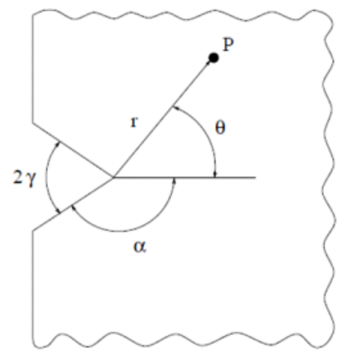
$$K_I := \lim_{r \to 0^+} \sigma_{22}\sqrt{2\pi r}$$

$$K_{II} := \lim_{r \to 0^+} \sigma_{12}\sqrt{2\pi r}$$

Second example with an exact solution

V notch Problem

$\alpha = \pi$: $\gamma = 0$ → sharp crack problem



The geometry and the loading are best expressed in the polar coordinate system, so it's easiest to solve the problem in the polar coordinate system

Polar Coordinate $\quad \Delta\Delta\psi(r,\theta) = 0 \quad (1)$

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Let's choose a solution in the form

$$\psi(r,\theta) = r^{\lambda+1} F(\theta) \qquad (2)$$

Plug (2) in (1)

$$\boxed{\frac{\partial^4 F(\theta)}{\partial \theta^4} + 2(\lambda^2+1)\frac{\partial^2 F}{\partial \theta^2} + (\lambda^2-1)^2 F(\theta) = 0} \qquad (3)$$

$$F(\theta) = e^{m\theta}$$

$$\left[m^4 + 2(\lambda^2+1)m^2 + (\lambda^2-1)^2 \right] e^{m\theta} = 0 \rightarrow$$

$$\left[m^4 + 2(\lambda^2+1)m^2 + (\lambda^2-1)^2 \right] e = 0 \rightarrow$$

$$\left[m^2 + (1-\lambda)^2 \right]\left[m^2 + (1+\lambda)^2 \right] = 0$$

It has 4 solution $\quad m = \mp i(1-\lambda)$

$\quad m = \mp i(1+\lambda)$

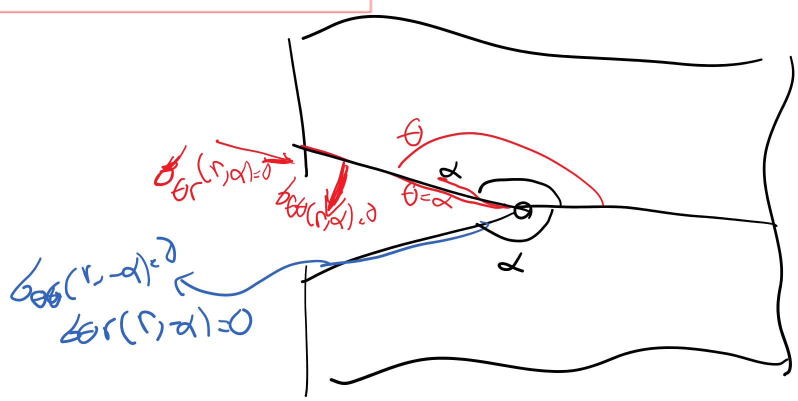
Eventually, $F(\theta)$, can be written as a linear combination of

$$\begin{cases} \cos(\lambda-1)\theta, & \sin(\lambda-1)\theta \\ \cos(\lambda+1)\theta, & \sin(\lambda+1)\theta \end{cases}$$

- Final form of stress function

$$\boxed{\Phi(r,\theta) = r^{\lambda+1} \underbrace{[A\cos(\lambda-1)\theta + B\cos(\lambda+1)\theta + C\sin(\lambda-1)\theta + D\sin(\lambda+1)\theta]}_{F(\theta,\lambda)}}$$

Find A to D



- Stress values

$$\sigma_{\theta\theta} = \frac{\partial^2\Phi}{\partial r^2} = r^{\lambda-1}\lambda(\lambda+1)F(\theta)$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) = r^{\lambda-1}[-\lambda F'(\theta)]$$

- Boundary conditions

$$\begin{aligned} \sigma_{\theta\theta}\big|_{\theta=\pm\alpha} &= 0 \\ \sigma_{r\theta}\big|_{\theta=\pm\alpha} &= 0 \end{aligned} \Rightarrow$$

$$F(\alpha) = F(-\alpha) = F'(\alpha) = F'(-\alpha) = 0 \Rightarrow$$

$$\boxed{\underbrace{\begin{bmatrix} \cos(\lambda-1)\alpha & \cos(\lambda+1)\alpha & 0 & 0 \\ \omega\sin(\lambda-1)\alpha & \sin(\lambda+1)\alpha & 0 & 0 \\ 0 & 0 & \sin(\lambda-1)\alpha & \sin(\lambda+1)\alpha \\ 0 & 0 & \omega\cos(\lambda-1)\alpha & \cos(\lambda+1)\alpha \end{bmatrix}}_{\det(\;)=0} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = 0}$$

$\omega = \dfrac{\lambda-1}{\lambda+1}$ $\qquad \det(\;) = 0$

$\lambda \neq 1$

- Eigenvalues and eigenvectors for nontrivial solutions

$$\boxed{\begin{aligned} \sin 2\lambda_n\alpha + \lambda_n \sin 2\alpha &= 0 \\ \sin 2\xi_n\alpha - \xi_n \sin 2\alpha &= 0 \end{aligned}} \begin{aligned} &\longleftarrow \text{Mode I} \\ &\longleftarrow \text{Mode II} \end{aligned}$$

135

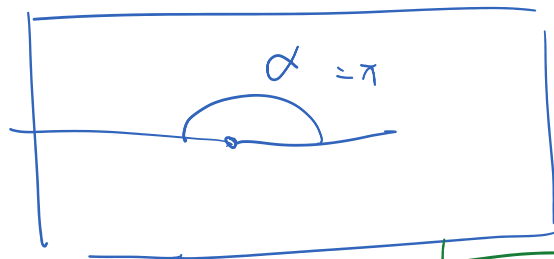
valid solutions for λ in $\psi(r,\theta) = r^{\lambda+1} F(\theta)$

In your HW or midterm you'll solve these equations and see how the solution becomes less singular as the notch angle (gamma) increase and eventually the singularity disappears at different notch opening angle for mode I and II.

How about a sharp crack example:

mode I $\quad \sin 2\lambda_n\pi + \lambda_n \cancel{\sin 2\pi} = 0$

mode II $\quad \sin 2\xi_n\pi - \xi_n \cancel{\sin 2\pi} = 0$



For both modes we have

$$\boxed{\sin 2\lambda\pi = 0} \longrightarrow \lambda_n = \frac{n}{2} = \left\{ \ldots, -\frac{5}{2}, -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots \right\}$$

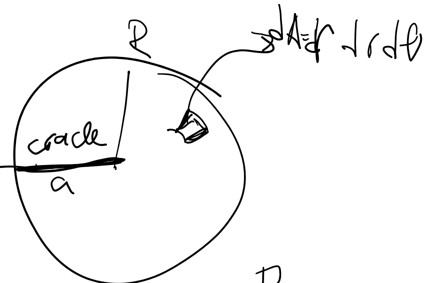
$\sigma \propto \frac{1}{\sqrt{r}} \qquad \sigma \propto \frac{1}{2}$

Not all of these solutions are valid from an energy perspective

$$\psi = r^{\lambda+1} F(\theta)$$

$$\longrightarrow \sigma's \propto r^{\lambda-1} F(\theta)$$

$$\underset{\text{strain energy}}{\mathcal{E}} = \int \frac{1}{2} \sigma : \varepsilon \, dA$$



$$= \frac{1}{2} \int_{r=0}^{R} \int_{\theta=0}^{2\pi} r^{\lambda-1} \, r^{\lambda-1} \overbrace{(r\,dr\,d\theta)}^{dA} \longrightarrow \int_0^R r^{2\lambda-1} dr \left(\int_0^{2\pi} (\;) \right)$$

$$\longrightarrow \frac{1}{2\lambda} r^{2\lambda} \Big|_0^R \propto \left[R^{2\lambda} - \underset{\varepsilon^{2\lambda}}{(0^+)^{2\lambda}} \right]$$

$\lambda > 0$

valid λ's are $\left\{ \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \right\}$

valid λ's are $\left\{ \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \right\}$

$\sigma \propto \frac{1}{\sqrt{r}}, \underbrace{(1)\ \sqrt{r}}_{\text{smaller compared to } \frac{1}{\sqrt{r}} \text{ term}}$

Leading term for stress solution is $\frac{1}{\sqrt{r}}$

calculate θ functions from det( ) = 0

- **First term** of stress expansion

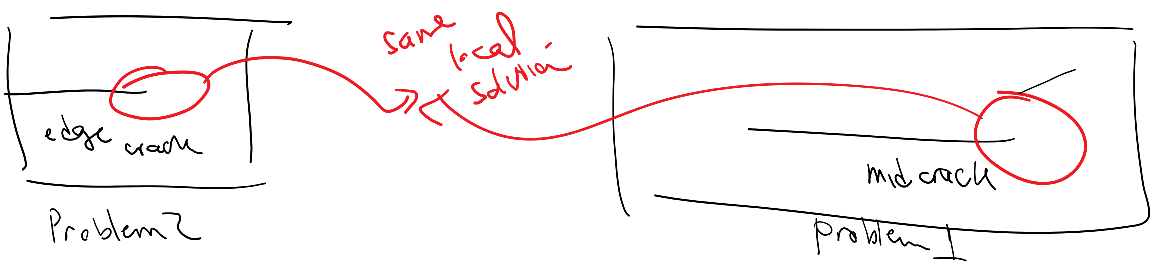
$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)]$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right)]$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)]$$

these match the other solution approach

- These terms are the leading stress solutions around the crack tip for modes I and II.
- The higher order terms (sqrt(r), r sqrt(r) , ...) are negligible compared to $\frac{1}{\sqrt{r}}$
- We notice that get the same exact solution form around the crack tip as with the other approach

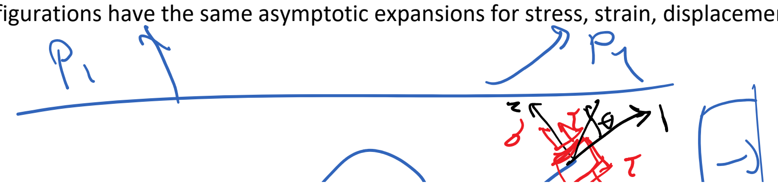


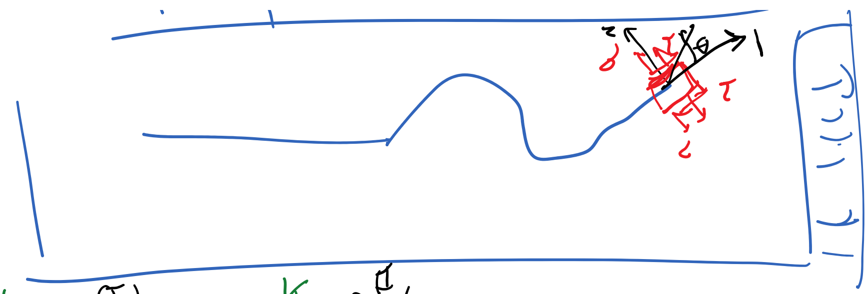
Why asymptotic solutions are useful?

Q: Do problem 1 & 2 have the global solution? NO

Q: Do they have the same local (asymptotic) solution around the crack tip?
Yes

Around the crack tip, all configurations have the same asymptotic expansions for stress, strain, displacements, ... expressed in terms of stress intensity factors





$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}} f^{(I)}_{rr}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f^{(II)}_{rr}(\theta)$$

$K_I = \sqrt{\pi a}\,\sigma$ for infinite mode I problem → here K_I (geometry, P_1, P_2, ...)

domain   crack path

as soon as we have K_I, K_{II}, K_{III} we have full characterization of stress, strain, displacement, etc, around the crack tip

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Displacement fields

$\sigma \longrightarrow \varepsilon = C^{-1}\sigma \longrightarrow$ integrate to get u

Asymptotic mode I displacement solution

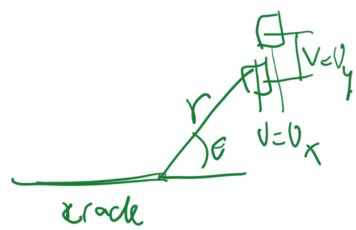
$$Z(z) = \frac{K_I}{\sqrt{2\pi r}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right) \qquad \xi^{-1/2}$$

$$Z(z) = \frac{K_I}{\sqrt{2\pi\xi}} \qquad \tilde{Z} = \int Z(z)dz$$

$$\tilde{Z}(z) = 2\frac{K_I}{\sqrt{2\pi}}\xi^{1/2} = 2K_I\sqrt{\frac{r}{2\pi}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

Recall

$$\boxed{\begin{aligned} 2\mu u &= \frac{\kappa-1}{2}\mathrm{Re}\,\tilde{Z} - y\,\mathrm{Im}Z \\ 2\mu v &= \frac{\kappa+1}{2}\mathrm{Im}\,\tilde{Z} - y\,\mathrm{Re}Z \end{aligned}}$$



$$z = \xi + a$$
$$\xi = re^{i\theta}$$

Displacement field

$$e^{-ix} = \cos x - i\sin x$$

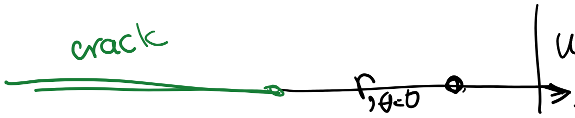
Displacement field

$$e^{-ix} = \cos x - i\sin x$$

$$\boxed{\begin{aligned} u &= \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left(\kappa - 1 + 2\sin^2\frac{\theta}{2}\right) \\ v &= \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(\kappa + 1 - 2\cos^2\frac{\theta}{2}\right) \end{aligned}}$$

Kolosov coef. κ

$$\kappa = \begin{cases} 3 - 4\nu & \text{plane strain} \\ \dfrac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$$



$$\boxed{u(r,\theta=0) = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\,(\kappa-1)}$$

$$V(r,\theta=0)=0$$

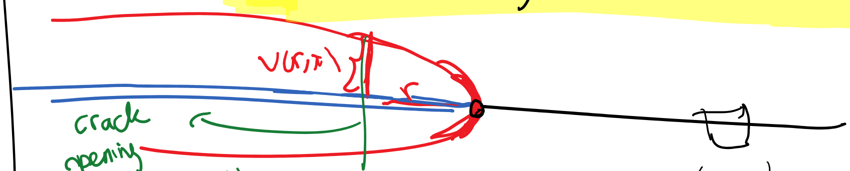
behind the crack

$\theta=\pi$

$$u = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\,\cancel{\cos\frac{\pi}{2}} \cdots = 0$$

$$V = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\,\sin\frac{\pi}{2}\left(\kappa+1-2\cancel{\cos^2\frac{\pi}{2}}\right)$$

$$V(r,\theta=90) = \frac{(\kappa+1)K_I}{2\mu}\sqrt{\frac{r}{2\pi}}$$



$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}}$$

crack opening is a parabolic curve