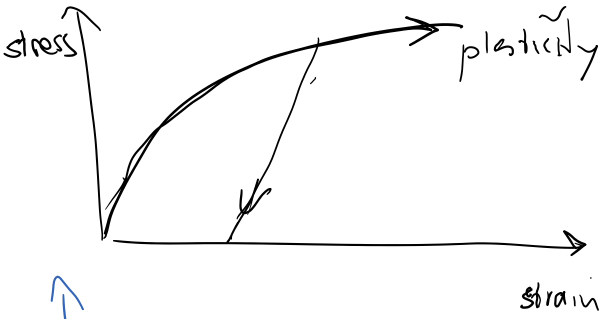
2022/10/11

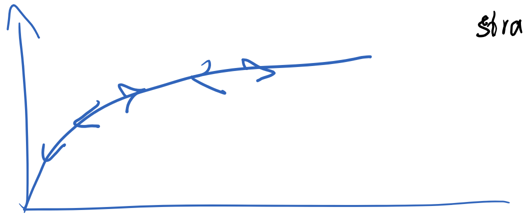
Tuesday, October 11, 2022 2:29 PM

J Integral (Rice 1958)

Modeling plasticity is difficult, how about approximating it with nonlinear elasticity theory



approximate this with nonlinear elasticity



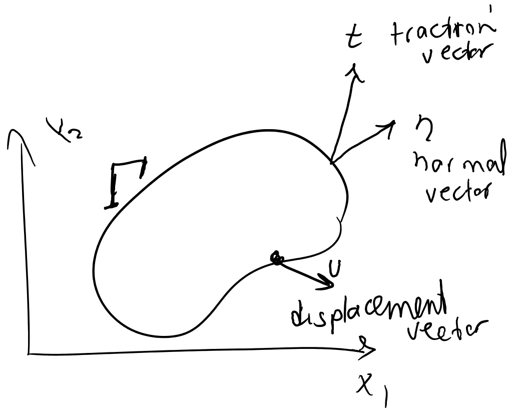
As long as we don't have significant unload events, the nonlinear elasticity model is accurate enough

J integral :

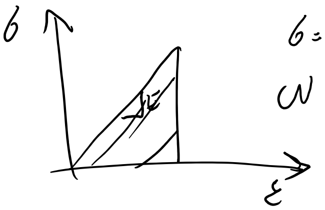
Starting point : Eshelby, Cherepanov 1967

$$k\in\{1,2\} : J_k := \int_\Gamma \left(W n_k - \underbrace{t_i \cdot \frac{\partial u_i}{\partial x_k}}_{\vec{t}\cdot\frac{\partial \vec{u}}{\partial x_k}} \right) d\Gamma$$

$W \downarrow$ strain energy density for an Elastic material

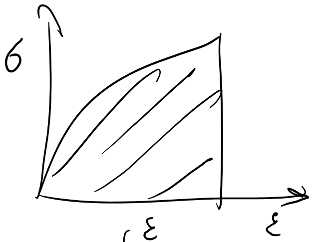


e.g. 1D elasticity



$\sigma = E\varepsilon$

$W = \frac{1}{2}E\varepsilon^2$



$\sigma = \frac{dW}{d\varepsilon} \Longleftrightarrow W = \int_0^\varepsilon \sigma(\tilde\varepsilon)d\tilde\varepsilon$

in 2D, 3D $\sigma = \frac{\partial W}{\partial \varepsilon}$

that is

$$\boxed{\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}}$$

Eshelby and Cherepanov showed that this integral is zero over a closed path for **smooth** solutions of **elastostatic** problems with **no body force**.

① ② ③

Equation of motion ② $\nabla\cdot\sigma + \underbrace{\rho b}_{\substack{0\\ ③}} = \underbrace{\rho\ddot{u}}_{\substack{0\\ ②}}$ → acceleration

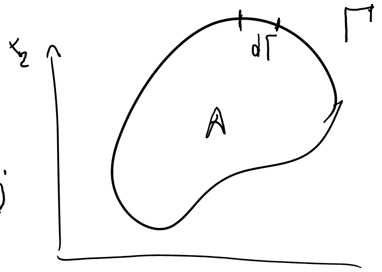
$$\boxed{\nabla\cdot\sigma = 0}$$

Proof:

$$J_k = \int_\Gamma \left(W(\varepsilon)\, n_k - \underbrace{t_i \frac{\partial u_i}{\partial x_k}}_{\sum_{i=1}^{2} t_i \frac{\partial u_i}{\partial x_k}} \right) d\Gamma$$

$t = \sigma n$

$t_i = \sum_{j=1}^{2} \sigma_{ij} n_j$



$$J_k = \int_\Gamma W(\varepsilon)\, n_k\, d\Gamma - \int_\Gamma \left(\overset{t_i}{\overline{\sigma_{ij} n_j}} \frac{\partial u_i}{\partial x_k} \right) d\Gamma$$

$\int_\Gamma f n_k = \underset{A}{\quad}{}_{k}\, dA$

To apply Gauss theorem the solution should be should enough (W & $\sigma \frac{\partial u}{\partial x_k}$ should be C^1) → condition ① above

Gauss theorem $\frac{\partial f}{\partial x_k}$ should be continuous in A

$f \in C^1(A)$

① $$J_k = \int_A W(\varepsilon)_{,k}\, dA - \int_A \left(\sigma_{ij} \frac{\partial u_i}{\partial x_k} \right)_{,j} dA$$

$$\frac{\partial W(\varepsilon)}{\partial x_k} = \underbrace{\frac{\partial W(\varepsilon)}{\partial \varepsilon_{mn}}}_{\sigma_{mn}} \frac{\partial \varepsilon_{mn}}{\partial x_k} = \frac{1}{2}\sigma_{mn}\left(u_{m,nk} + u_{n,mk} \right)$$

$\varepsilon_{mn} = \frac{1}{2}\left(u_{m,n} + u_{n,m} \right)$

Since $\sigma_{mn} = \sigma_{nm}$ →

② $$\frac{\partial W(\varepsilon)}{\partial x_k} = \frac{1}{2}\sigma_{mn} u_{m,nk} + \frac{1}{2}\underset{\sigma_{mn}}{\sigma_{nm}} u_{m,nk} = \sigma_{mn} u_{m,nk}$$

③ $$\left(\sigma_{ij} \frac{\partial u_i}{\partial x_k} \right)_{,j} = \underbrace{\sigma_{ij,j}}_{(\nabla\cdot\sigma)_i = 0} \frac{\partial u_i}{\partial x_k} + \sigma_{ij} \frac{\partial^2 u_i}{\partial x_j \partial x_k} = \sigma_{ij} u_{i,jk}$$

"no body force"
"no acceleration"

①②③ → $$J_k = \int_A \left(\sigma_{mn} u_{m,nk} - \underset{m\ n}{\sigma_{ij}} u_{i,jk} \right) dA$$

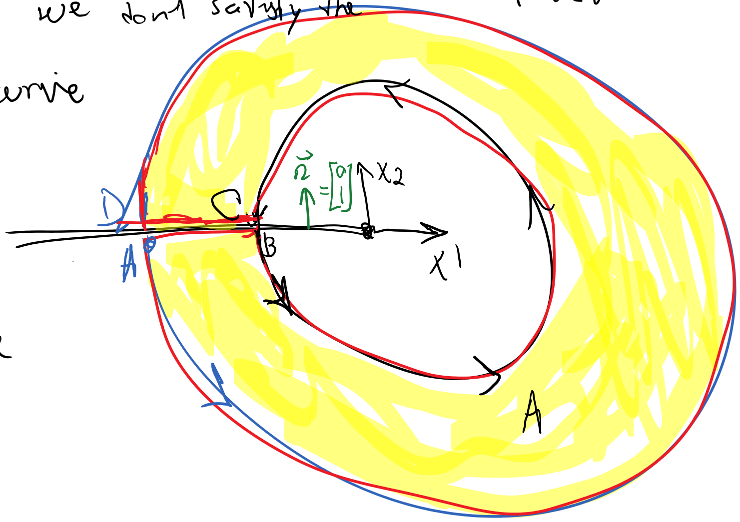
A $\quad$ $\downarrow\downarrow$ $m\ n$

$$= \int_A \left(\sigma_{mn} u_{m,nk} - \sigma_{mn} u_{m,nk}\right) dA = 0$$

Show that $\boxed{J1}$ integral takes the same value over any curve around the crack tip

$$\begin{cases} b = 0 \\ \ddot{u} = 0 \end{cases}$$

$J_{BC} = \int_B \text{integrand} \neq 0$ because we dont satisfy the smoothness requirement for this curve



$$J_{BC} = J_{AD}$$

for any path around the crack we get the same value

in A solution is smooth

$$J_{ABCDAB} = 0$$

$$= J_{BC} + \cancel{J_{CD}} + J_{DA} + \cancel{J_{AB}} = 0$$

$$J_{CD} = \int \left[W(\varepsilon) \underset{0}{n_1} - \underset{0}{\vec{t}} \frac{\partial \vec{u}}{\partial x_1} \right] d\Gamma$$

$n_1 = 0$
on the crack surface

we're considering J_1
$\rightarrow x_k \rightarrow x_1$

$\textcircled{t}$

we make the assumption that the crack surface is traction

$$J_{CD} = 0$$

similarly

$$J_{AB} = 0$$

$$J_{BC} + J_{DA} = 0$$

$$J_{BC} = -J_{DA} = J_{AD}$$

$$\boxed{J_{BC} = J_{AD}}$$

As long as the crack surface is traction free we can calculate J around the crack along any path

Side note: For J2 top and bottom crack surface integrals cancel out if (C and B) and (D and A) coincide and we reach the same conclusion

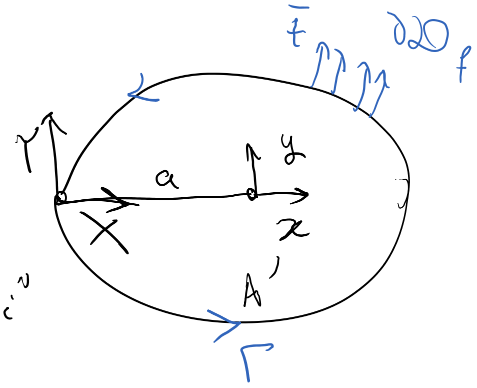
---

J Rice $\quad J = J_1 = G$ $\quad$ J is equal to energy release rate

$$G = -\frac{\partial \Pi}{\partial \underset{\text{crack surface}}{A}} = -\frac{\partial \Pi}{\partial \underset{\text{width}}{B}\underset{\text{crack length}}{a}}$$

B = 1 here

(X,Y) fixed coordinate
(x,y) move with the crack tip



$$\Pi = \underset{\text{internal energy}}{U_e} - \underset{\text{external work}}{W}$$

no kinetic energy "quasi-static"

$$U_e = \int_{A'} \underbrace{\mathcal{W}(\varepsilon)}_{\text{internal energy density}} dV$$

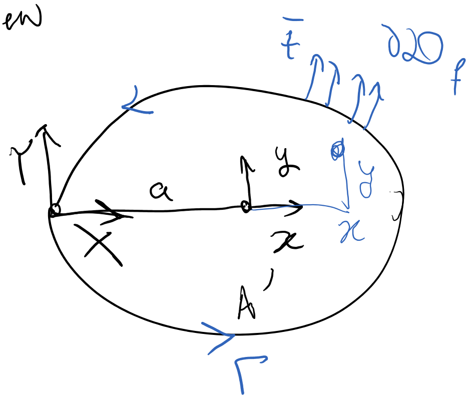
$$W = \int_{A'} u \cancel{\rho b}\, dV + \int_{\partial D_f} u \bar{t}\, d\Gamma$$

$\cancel{\rho b} \to 0$ "no body force"

$$G = \frac{-d\Pi}{da} = -\frac{d}{da}\left(\int_{A'} \mathcal{W}(\varepsilon)\, dV - \int_{\partial D_f} u \bar{t}\, d\Gamma\right)$$

$$= \int_{A'} \frac{d\mathcal{W}(\varepsilon)}{da}\, dV + \int \frac{du}{da} \vec{t}\, d\Gamma$$

skipped a few steps



http://rezaabedi.com/wp-content/uploads/Courses/FractureMechanics/J=G.pdf

$\left.\frac{df}{da}\right|_{X \text{ is fixed}}$

Since the J integral is computed in the coordinate system attached to the crack tip, we need to change the coordinate from (X, Y) to (x, y) for computing the

∂f

$da \quad \Big| \; X$ is fixed

$\dfrac{\partial f}{\partial a} \Big| \; x$ is fixed

Since the J integral is computed in the coordinate system attached to the crack tip, we need to change the coordinate from (X, Y) to (x, y) for computing the derivatives

$$X = a + x \quad + \cancel{x_c}$$

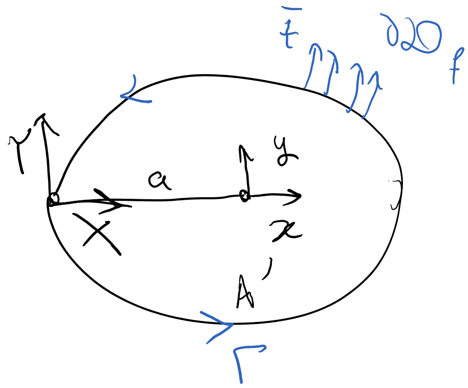
$$x = X - a$$

$$\frac{d f(x,a)}{da}\Big|_{X\text{-fixed}} = \frac{\partial f(x,a)}{\partial x} \underset{(-1)}{\frac{\partial \overset{X-a}{x}}{\partial a}} + \frac{\partial f}{\partial a} \underset{(1)}{\frac{\partial a}{\partial a}}$$

$$\boxed{\frac{df}{da}\Big|_{X\text{-fixed}} = \underbrace{\frac{\partial f}{\partial a} - \frac{\partial f}{\partial x}}_{x\text{-coordinate}}}$$

$$G = -\int_{A'} \frac{dW}{da}\, dA + \int_{\Gamma} \frac{du}{da}\, \vec{t}\, ds$$

$$\frac{dW}{da} \rightarrow \frac{\partial W}{\partial a} - \frac{\partial W}{\partial x} \qquad \frac{du}{da} \rightarrow \frac{\partial u}{\partial a} - \frac{\partial u}{\partial x}$$



$$G = -\int_{A'} \frac{\partial W}{\partial x}\, dA + \int_{\Gamma} \vec{t}\, \frac{\partial \vec{u}}{\partial x}\, ds$$

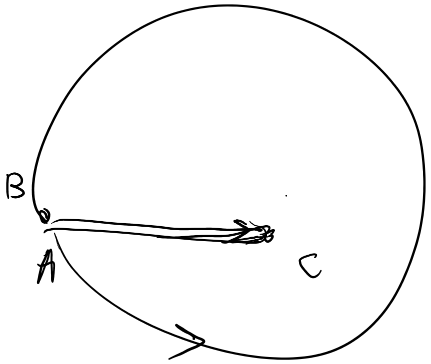
$$+ \left(\int_{A'} \frac{\partial W}{\partial a}\, dA - \int_{\Gamma} \vec{t}\, \frac{\partial \vec{u}}{\partial a} \right) = 0$$

similar proof to the one we had $\int_{\Gamma} (\;\;)\, ds = 0$ in the beginning of the class

$$G = -\int_{A'} \underbrace{\frac{\partial W}{\partial x}\, dA}_{\text{use divergence theorem}} + \int_{\Gamma} t\, \frac{\partial u}{\partial x}\, ds$$

$$= \int_{\Gamma} -W n_1\, ds + \int_{\Gamma} \vec{t}\, \frac{\partial \vec{u}}{\partial x}\, ds$$

$$= \int_{\Gamma} \underbrace{\left(-W n_1 + \vec{t}\, \frac{\partial u}{\partial x}\right)}_{J} ds$$



$$G = \int J + \int J + \int J$$

$$G = \overset{1}{\int_{AB} J} + \overset{+}{\int_{BC} J} + \int_{CA} J \; -$$

$\int_{BC} \to 0$ again because $\left(\begin{array}{l} \vec{t} = 0 \text{ on crack surfaces} \\ n_1 = 0 \text{ on crack surfaces} \end{array} \right)$

$$\boxed{G = \int_A^B \left(-W n_1 + \vec{t} \frac{\partial u}{\partial x} \right) ds = J}$$

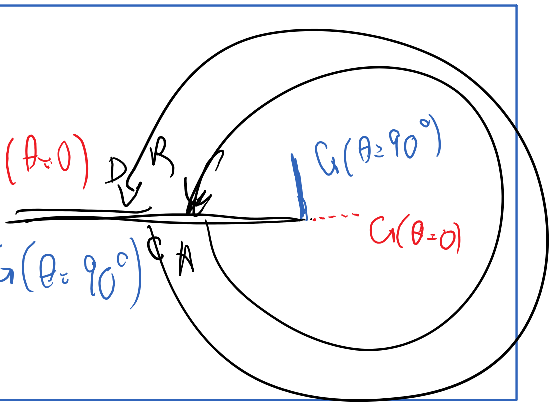
Energy release rate is equal to $J (= J_1)$

$$\boxed{\begin{array}{l} G(\theta=0) = J_1 \\ G(\theta = \frac{\pi}{2}) = J_2 \end{array}}$$

Summary:

$$J_1 = \int_A^B \left(-W\, dy + \vec{t} \frac{\partial \vec{u}}{\partial x} \right) ds = G(\theta=0)$$

$$J_2 = \int_A^B \left(W\, dx + \vec{t} \frac{\partial \vec{u}}{\partial y} \right) ds = G(\theta = 90^\circ)$$



A → B , C → D

Relate Ks & Js:

Prev. we had only G but K_I, K_{II}

$$K_I, K_{II} \longrightarrow G$$
$$\cancel{\longleftarrow}$$

$$J_1 \;\&\; J_2 \rightleftarrows K_I, K_{II}$$

next time

$J_1 \downarrow G(\theta=0)$ $J_2 \downarrow G(90^\circ)$