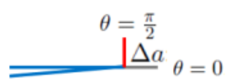
2022/10/13

Thursday, October 13, 2022 2:21 PM

In fact both J1 (J) and J2 are related to SIFs:

$$J_1 = \int_\Gamma \left(w dy - \mathbf{t}\frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)$$
$$J_2 = \int_\Gamma \left(-w dx - \mathbf{t}\frac{\partial \mathbf{u}}{\partial y} d\Gamma \right)$$

J1 & J2: crack advance for (θ = 0, 90) degrees



$$J = J_1 - iJ_2 \quad \text{Hellen and Blackburn (1975)}$$
$$= \frac{(1+\nu)(1+\kappa)}{4E}(K_I^2 + K_{II}^2 + 2iK_I K_{II})$$

$$J_1 = \frac{K_I^2 + K_{II}^2}{E'} \qquad J_2 = -\frac{K_I K_{II}}{E'} \rightarrow K_{II} = \frac{-J_2 E'}{K_I}$$

$$K_I^2 + \left(-\frac{J_2 E'}{K_I}\right)^2 = J_1 E' \qquad \times K_I^2$$

$$K_I^4 - (J_1 E')K_I^2 + (J_2 E')^2 = 0$$

$$z^2 - (J_1 E')z + (J_2 E')^2 = 0 \qquad z = K_I^2$$

The solutions will be $K_I^2 = a^2$ or b^2 →

$K_I = \mp a$, $K_{II} = \mp b$ $\qquad$ $K_I = \mp b$, $K_{II} = \mp a$

We'll choose the right values and signs based on the loading and direction of loading.

$$\boxed{J_1 = \frac{K_I^2 + K_{II}^2}{E'} \qquad J_2 = \frac{-2K_I K_{II}}{E'}}$$

$$E' = \begin{cases} E & \text{plane strain} \\ \frac{E}{1-\nu^2} & \text{plane stress} \end{cases}$$

Note that if $K_I = a, K_{II} = b$ is a solution the general solution is:
$K_I = \pm a, K_{II} = \pm b$ and $K_I = \pm b, K_{II} = \pm a$

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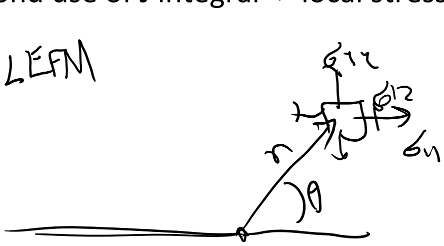
Second use of J integral -> local stress distribution

LEFM



$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} \Sigma_I^{ij}(\theta) + \cdots \text{mode II}$$

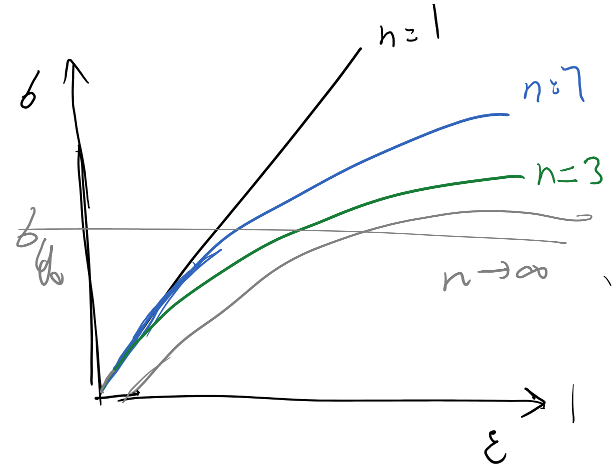
PFM

$$\sigma_{ij} \propto J^\alpha r^\beta$$

?

Do we have an asymptotic solution of this form

5.3.5 Plastic crack tip fields:
Hutchinson, Rice, and Rosengren (HRR) solution



Ramberg - Osgood model

$$\frac{\varepsilon}{\varepsilon_{y_0}} = \frac{\sigma}{\sigma_{y_0}} + \alpha\left(\frac{\sigma}{\sigma_{y_0}}\right)^n$$

n=1 → similar to ε^e $\qquad$ $\alpha\left(\frac{\sigma}{\sigma_{y_0}}\right)^n \approx \varepsilon^p$

Compare with plasticity model

$$\varepsilon = \varepsilon^e + \varepsilon^p$$

$$\frac{\varepsilon}{\varepsilon_{y_0}} = \left(\frac{\sigma}{\sigma_{y_0}}\right) + \underbrace{\alpha\left(\frac{\sigma}{\sigma_{y_0}}\right)^n}_{\text{very large around the crack tip}}$$

around the crack tip

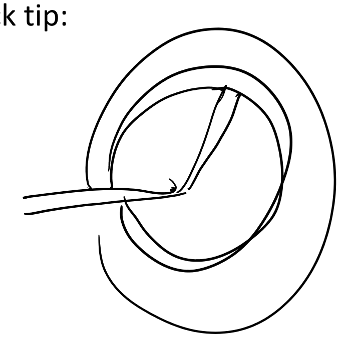
$$\frac{\varepsilon}{\varepsilon_{y_0}} \approx \alpha\left(\frac{\sigma}{\sigma_{y_0}}\right)^n \qquad \boxed{\varepsilon \propto \sigma^n} \quad (1) \qquad \text{around the crack tip}$$

We are looking for the powers of singularity for strain and stress around the crack tip:

(2) $\sigma = \frac{c_1}{r^x} \qquad \varepsilon = \frac{c_2}{r^y}$

From energy argument $\qquad \mathcal{E} = \int_0^R \mathcal{W}(\varepsilon)\, dA$

$\mathcal{W}(\varepsilon) \to \varepsilon : \sigma$, $\quad dA \to r\,d\theta\,dr$



$$\int \frac{r}{r^y} \cdot \frac{1}{r^x}$$

$$r^{1-x-y}$$

$1 - x - y \geq 0$

$\underline{x + y \leq 1}$ $\quad$ In fact $\underline{x + y = 1}$ $\quad \to (n+1)y = 1 \qquad y = \frac{1}{1+n}$

(1),(2) $\quad \varepsilon \propto \sigma^n$

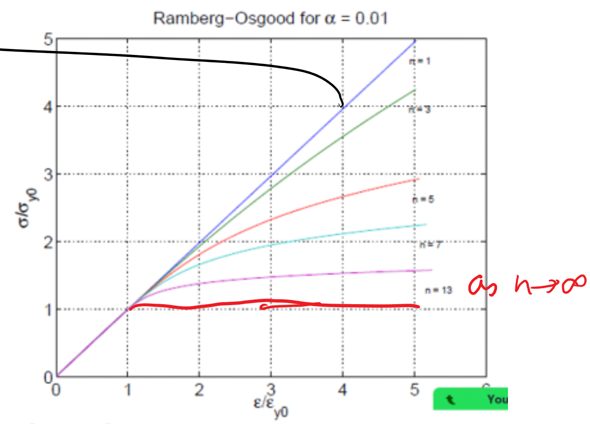
(1), (2) $\varepsilon \propto \sigma^n$

$\dfrac{c_2}{r^x} \propto \left(\dfrac{c_1}{r^y}\right)^n \longrightarrow x = ny$ $\Big\} \longrightarrow (n+1)y = 1$ $\quad y = \dfrac{1}{1+n}$, $\quad x = \dfrac{n}{1+n}$

we expect to have the following dependence on r around the crack tip

$$\sigma \propto \frac{1}{r^{\frac{1}{n+1}}} \qquad \varepsilon \propto \frac{1}{r^{n/n+1}}$$

$n = 1 \rightarrow$ LEFM (linear elastic) $\leftarrow$



$$\sigma \propto \frac{1}{\sqrt{r}} \qquad \varepsilon \propto \frac{1}{\sqrt{r}}$$

we had this before

$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} \Sigma_{ij}^{I}(\theta) + \ldots$$

$$\varepsilon_{ij} = D_{ijkl}\,\sigma_{kl} \propto \frac{1}{\sqrt{r}}$$

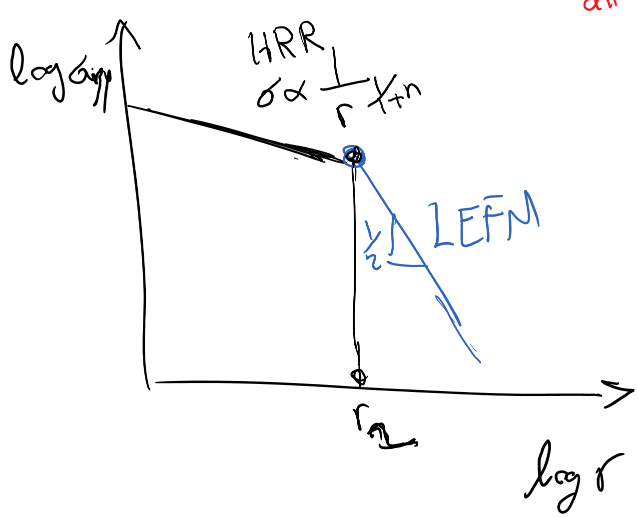
How about the limit as $n \to \infty$ (approximate an elastic-perfectly plastic model)

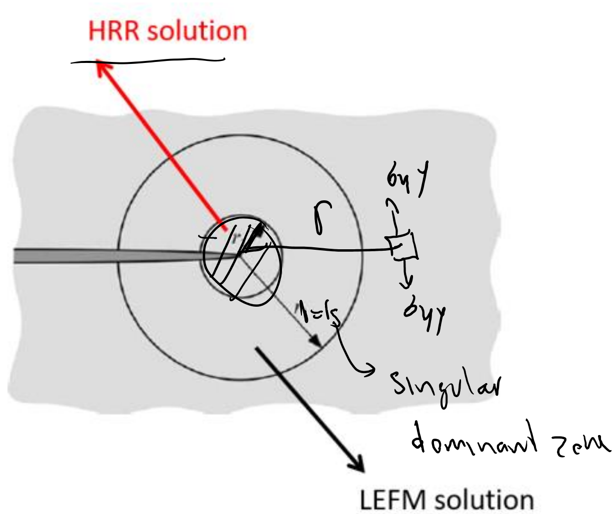
$\sigma \rightarrow \dfrac{1}{r^0} \rightarrow$ const no longer singular in the limit of elastic perfectly plastic.

σ is almost bounded by σ_{y_0} (as if we have yield stress)

$\varepsilon \propto \dfrac{1}{r^{n/n+1}} \longrightarrow \dfrac{1}{r}$

all singularity goes to strain





- This model still predicts that stress is singular close to the crack tip, but the power of singularity for stress is lower.
- Only predicts bounded stress for the limit of n -> infinity which corresponds to elastic-perfectly plastic model.

Final form of HRR solution

plays the role of K

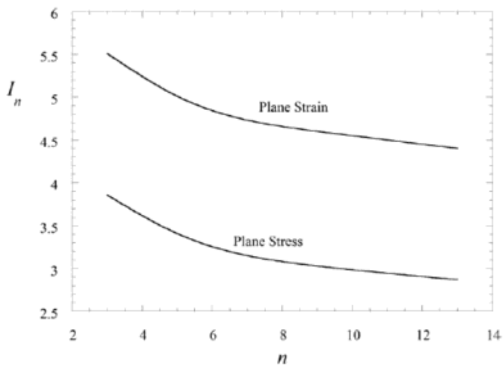
expressed in terms of G it has the power of 0.5

$$\sigma_{ij} = \sigma_0 \left(\frac{EJ}{\alpha \sigma_0^2 I_n r} \right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(n, \theta)$$

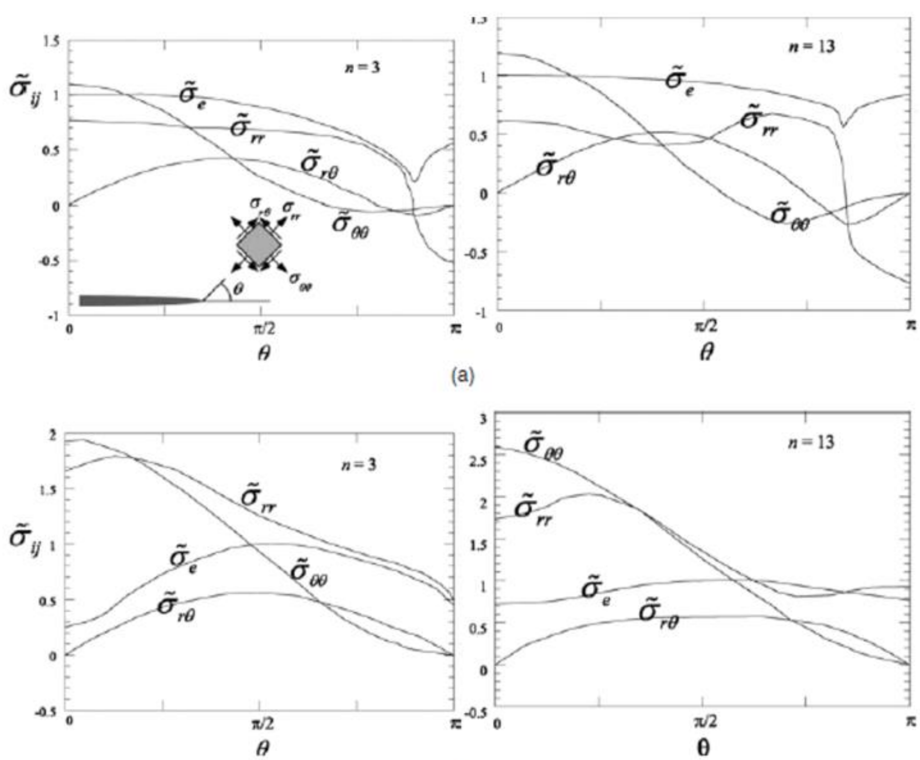
$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} \Sigma_{ij}^{I}(\theta)$$

$$\epsilon_{ij} = \frac{\alpha \sigma_0}{E} \left(\frac{EJ}{\alpha \sigma_0^2 I_n r} \right)^{\frac{n}{n+1}} \tilde{\epsilon}_{ij}(n, \theta)$$

terms in green show dependency on power n



HRR solution:Angular functions

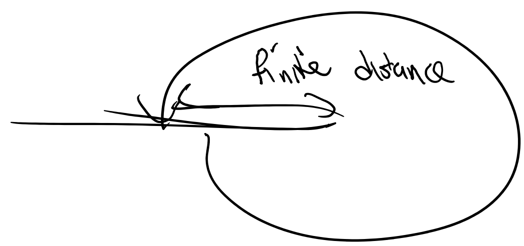


Summary:
J plays the role of G and K's in PFM:

	LEFM	PFM
Global perspective energy	$G = -\frac{d\Pi}{dA}$ E.R.R	$G = J$ even when LEFM is not applicable
Local perspective stress/strain solutions	$\sigma \propto \frac{K}{\sqrt{r}} \propto \frac{\sqrt{GE}}{\sqrt{r}}$	$\sigma \propto \left(\frac{J}{r}\right)^{\frac{1}{1+n}}$ $\varepsilon \propto \left(\frac{J}{r}\right)^{n/(n+1)}$

Energy release rate of J integral: Assumptions

1. Homogeneous body
2. Linear or non-linear elastic solid → approximates plasticity
3. No inertia, or body forces; no initial stresses $\nabla\cdot\sigma + \cancel{\rho b} = \cancel{\rho \ddot{u}}$
4. No thermal loading
5. 2-D stress and deformation field
6. Plane stress or plane strain
7. ~~Mode I loading~~
8. Stress free crack



We can extend the way we compute the J integral so that none of these constraints become relevant (we will do this when covering the computational methods for computing J)

Generalization of J integral

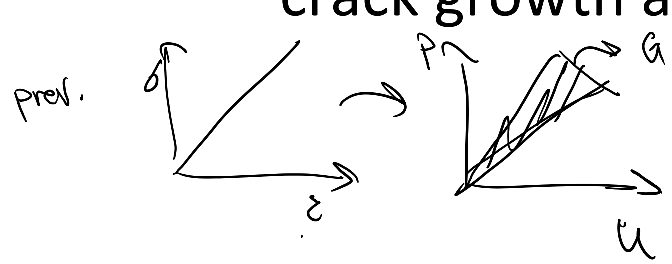
- Dynamic loading
- Surface tractions on crack surfaces
- Body force
- Initial strains (e.g. thermal loading)
- Initial stress from pore pressures

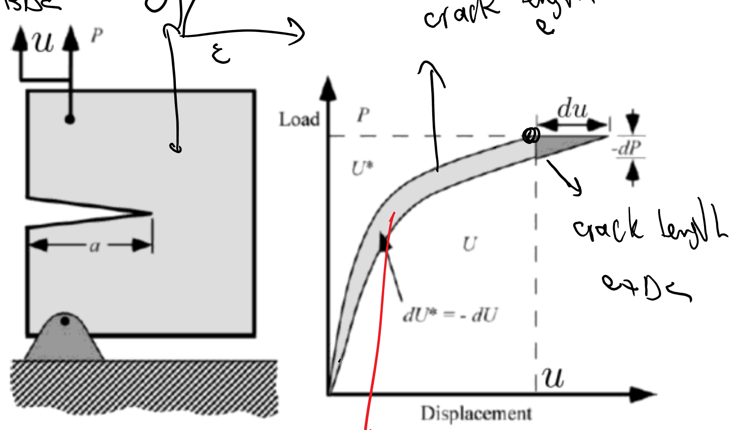
cf. Saouma 13.11 & 13.12 for details

Linear -> nonlinear **elasticity**, extending the P-delta system analysis

5.3.4. Energy Release Rate, crack growth and R curves

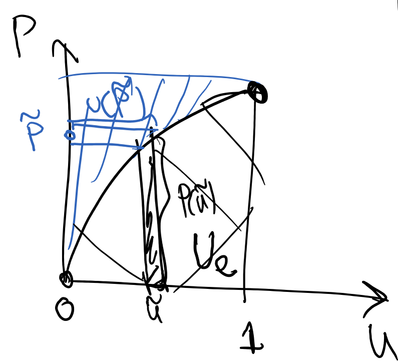
prev.





$$G = \frac{area}{B\,\Delta a}$$

Definitions

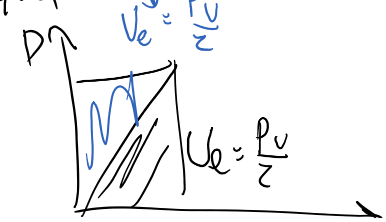


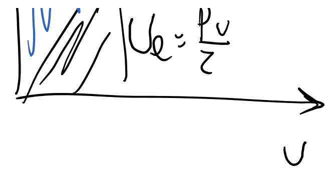
$$U_e = W_{el} = \int_0^u P(\tilde{u})\, d\tilde{u}$$

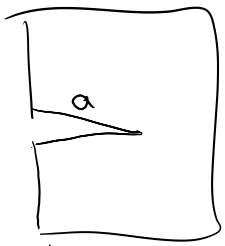
for fixed crack length (if there is a crack)

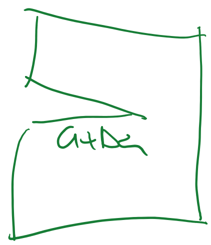
$$\boxed{U_e^* = \int_0^P u(\tilde{P})\, d\tilde{P} = Pu - U_e}$$

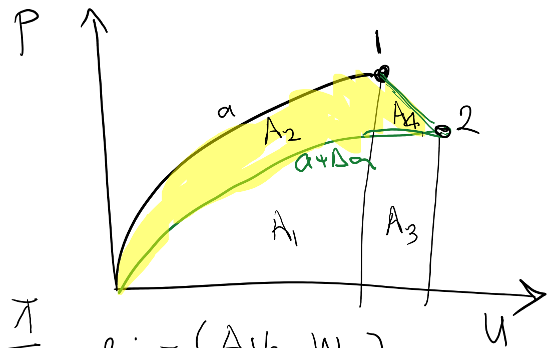
For linear elasticity the two are equal











want to show

$$G = \lim_{\Delta a \to 0} \frac{\text{shaded area}}{\Delta a \cdot B}$$

$$G = -\frac{d\Pi}{dA} = -\frac{d\Pi}{B\,da} = \lim_{\Delta a \to 0} -\frac{(\Delta U_e - W_{12})}{B\,\Delta a}$$

$$\Delta U_e = U_e(a+\Delta a) - U_e(a) = (A_1 + A_3) - (A_1 + A_2) = A_3 - A_2$$

$$U_e(a+\Delta a) = A_1 + A_3$$

$$U_e(a) = A_1 + A_2$$

$$W_{12} = \int_1^2 P\,du = A_3 + A_4$$

$$G = \lim -\frac{(A_3 - A_2 - (A_3 + A_4))}{B\,\Delta a}$$

$$\to \lim_{\Delta a \to 0} \frac{A_2 + A_4}{B\,\Delta a}$$

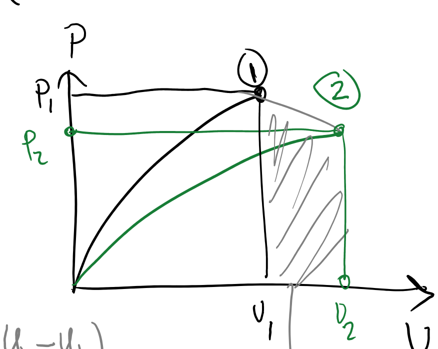
$$\boxed{\text{linear \& nonlinear elasticity} \qquad G = \lim_{\Delta a \to 0} \frac{\text{shaded area}}{B\,\Delta a}}$$

Do we have equations similar to

$$G = \frac{P^2}{2B}\frac{dC}{da} = -\frac{u^2}{2B}\frac{dK}{da}$$

compliance $C = \frac{u}{P}$

stiffness $K = \frac{P}{u}$



Recall

$$G = \lim_{\Delta a \to 0} -\frac{\{\overbrace{(U_{e_2} - U_{e_1})}^{\Delta U_e} - W_{12}\}}{B\,\Delta a}$$

$$W_{12} \approx \left(\frac{P_1 + P_2}{2}\right)(u_2 - u_1)$$

$$= \frac{1}{B}\lim_{\Delta a \to 0} -\underbrace{\left(\frac{U_e(a+\Delta a) - U_e(a)}{\Delta a}\right)} + \frac{1}{B}\lim_{\Delta a \to 0} \frac{\left(\frac{P_1+P_2}{2}\right)(u_2 - u_1)}{\Delta a}$$

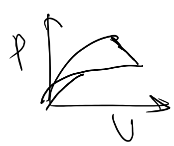
$\Delta a \to 0$ $\quad$ $\Delta a \to 0$ $\qquad W_{12} \approx \left(\frac{P_1+P_2}{2}\right)(u_2-u_1)$

$$G = \frac{-1}{B}\frac{dU_e(a)}{da} + \frac{1}{B}\lim_{\Delta a \to 0}\left(\frac{P(a)+P(a+\Delta a)}{2}\right)\left(\frac{u(a+\Delta a)-u(a)}{\Delta a}\right)$$

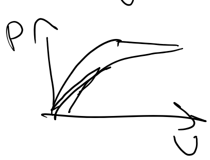
$$f'(x) = \lim_{\Delta x \to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$G = -\frac{1}{B}\frac{dU_e}{da} + \frac{P}{B}\cdot\frac{du}{da}$$

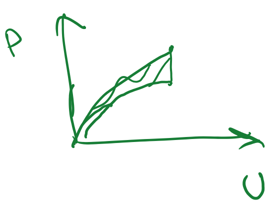
General formula for G



$$G = \frac{1}{B}\left(-\frac{dU_e}{da} + P\frac{du}{da}\right)$$

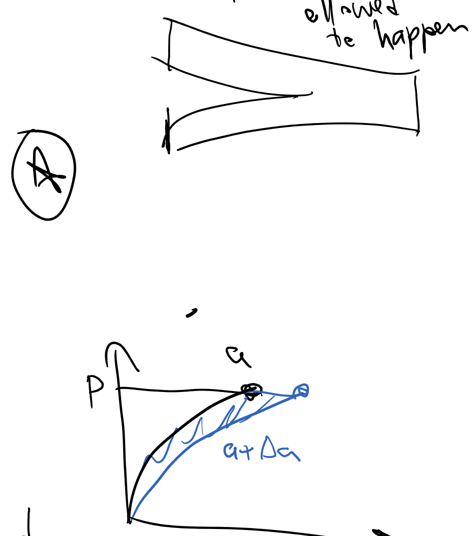


$$= \frac{1}{B}\frac{dU_e^{*}}{da} \quad \text{dead load}$$



$$= \frac{-1}{B}\frac{dU_e}{da} \quad \text{fixed grip}$$

plasticity is allowed to happen



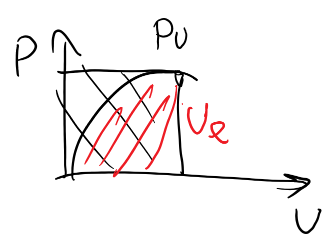
Special case 1: dead load (P is fixed)

$$= \frac{1}{B}\left(-\frac{dU_e}{da} + P\frac{du}{da}\right)$$

take P inside $\frac{d}{da}$ because it's fixed

$$= \frac{1}{B}\left(-\frac{dU_e}{da} + \frac{dPu}{da}\right) = \frac{1}{B}\frac{d(Pu - U_e)}{da}$$

$$= \frac{1}{B}\frac{dU_e^{*}}{da} \quad \text{dead load}$$

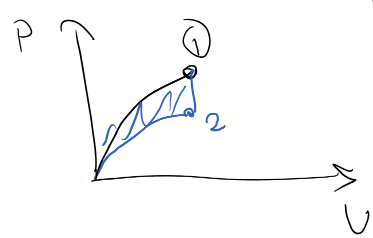


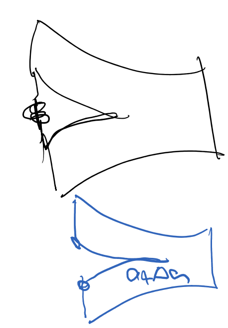
Special case 2: fixed grip

$$G = \frac{1}{B}\left(-\frac{dU_e}{da} + P\cancel{\frac{du}{da}}^{\,0}\right)$$

u is fixed $\rightarrow \frac{du}{da} = 0$

$$G = \frac{-1}{B}\frac{dU_e}{da}$$

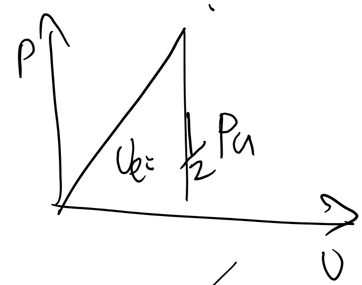




Does (*) match linear results for a linear material?

$$G = \frac{1}{B}\left(-\frac{dU_e}{da} + P\frac{du}{da}\right)$$

$$U_e = \frac{1}{2}Pu = \frac{1}{2}(\underbrace{ku}_{P})u = \frac{1}{2}ku^2$$



$$G = \frac{1}{B}\left(-\frac{d\,\frac{1}{2}ku^2}{da} + P\frac{du}{da}\right) = \frac{1}{B}\left(-\frac{1}{2}\frac{dk}{da}\cdot u^2 \cancel{- \underbrace{ku}_{P}\frac{du}{da} + P\frac{du}{da}}\right)$$

$$= -\frac{u^2}{2B}\frac{dK}{da}$$ matches our linear formula

$$= \frac{P^2}{2B}\frac{dC}{da}$$