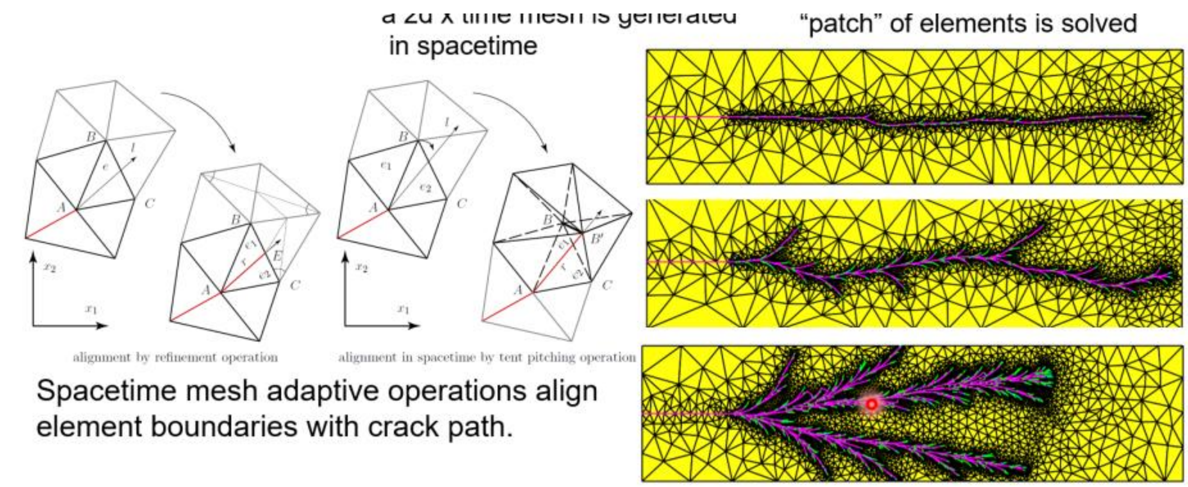
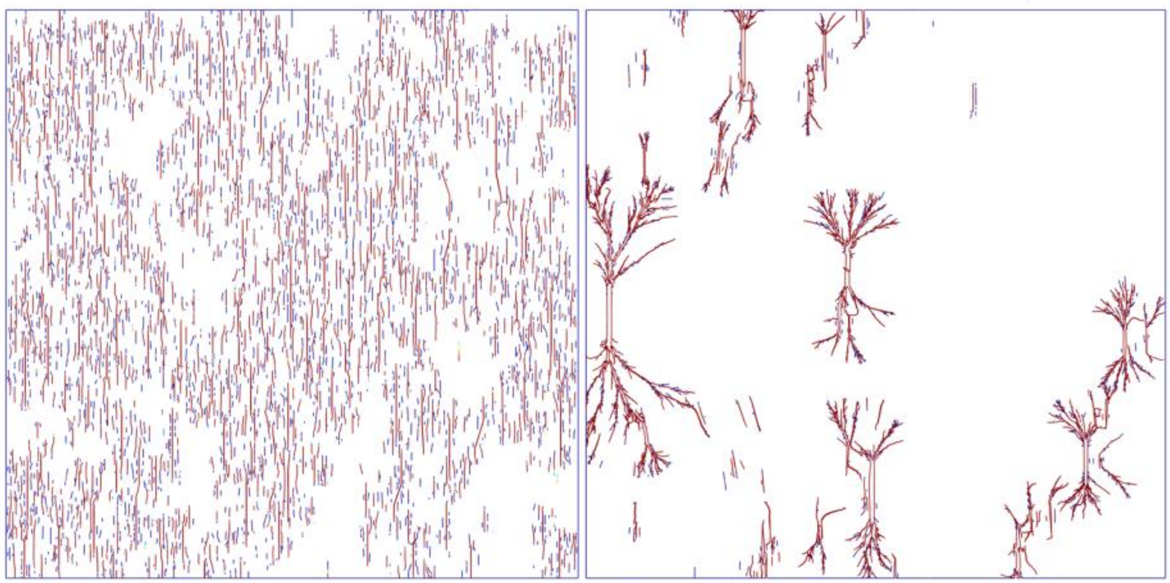
2022/10/25

Tuesday, October 25, 2022 2:28 PM

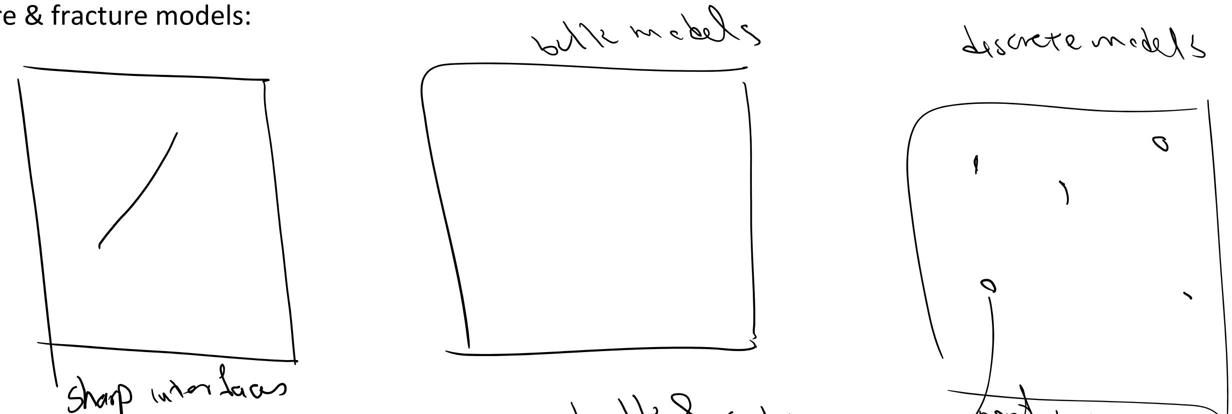




Homogeneous $\bar{s}$ **Random $\bar{s}$, SVE1x1 sampling**

Summary:

Failure & fracture models:



sharp interfaces
- LEFM
- Cohesive models

- bulk & continuum damage model

$\lambda = (1-D) G_1$

point particles
- peridynamics

— LEFM
— Cohesive models

Challenge
modeling the geometry

focus of this fracture course

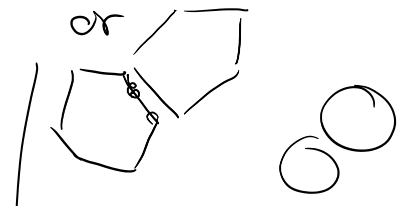
damage model

$\sigma = (1-D)\,\sigma_{el}$

— phase field
regularized LEFM

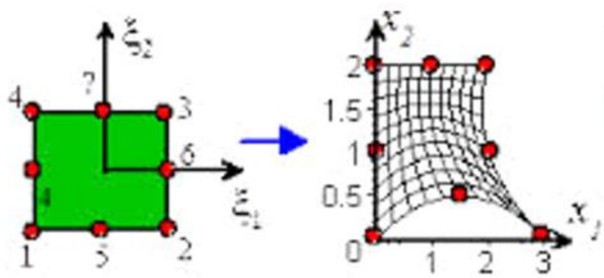
— peridynamics

or



Discrete Element method
(DEM)
geomechanics

Isoparametric Elements



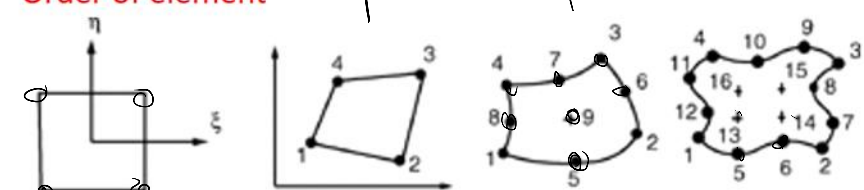
parent element

Actual element

- Geometry is mapped from a parent element to the actual element
- The same interpolation is used for geometry mapping and FEM solution (in the figure 2nd order shape functions are used for solution and geometry)
- Geometry map and solution are expressed in terms of ξ

Order of element

$p = 1$ $p = 2$ $p = 3$



Interpolation space
(Same number of nodes based on the order)

a) Linear b) Quadratic c) Cubic

Physical space

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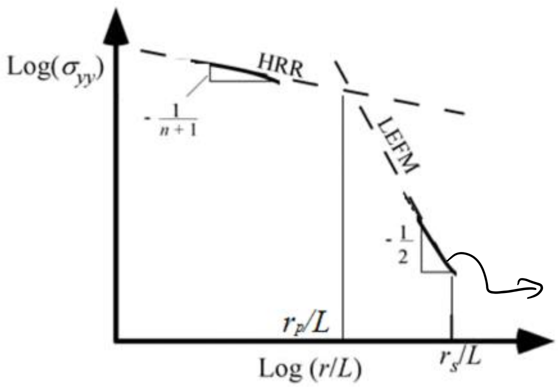
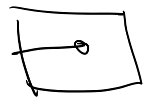
Singular crack tip solutions



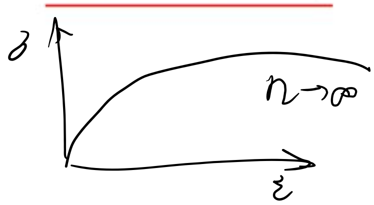
$$\sigma_{ij} = \sigma_0 \left(\frac{EJ}{\alpha \sigma_0^2 I_n r} \right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(n,\theta)$$

$$\epsilon_{ij} = \frac{\alpha \sigma_0}{E} \left(\frac{EJ}{\alpha \sigma_0^2 I_n r} \right)^{\frac{n}{n+1}} \tilde{\epsilon}_{ij}(n,\theta)$$



ε & $\sigma \propto \frac{1}{\sqrt{r}}$

- NLFM (PFM): For HRR solution stress $\frac{1}{r^{\frac{1}{n+1}}}$ and strain $\frac{1}{r^{\frac{n}{n+1}}}$ are still singular $\Rightarrow$
  - for elastic-perfectly plastic ($n \to \infty$) stress is bounded and strain is $\frac{1}{r}$ singular

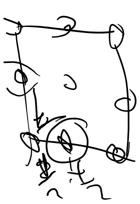


$\varepsilon \to$ elastic perfectly plastic

LEFM

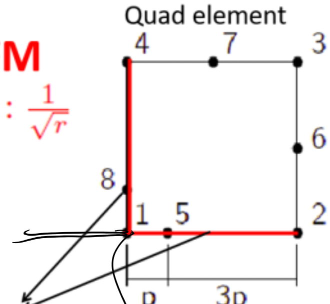
Isoparametric singular elements

$P=2$



- **LEFM**

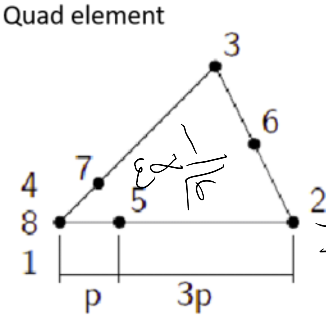
$\epsilon, \sigma : \frac{1}{\sqrt{r}}$



singular form $\frac{1}{\sqrt{r}}$ only along these lines

NOT recommended

want this point to be a crack tip

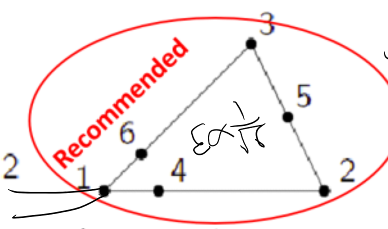


Improvement:

- $\frac{1}{\sqrt{r}}$ from inside all element

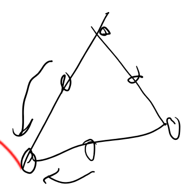
Problem

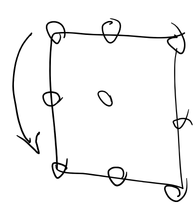
- Solution inaccuracy and sensitivity when opposite edge 3-6-2 is curved



Improvement:

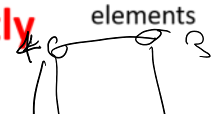
- Better accuracy and less mesh sensitivity

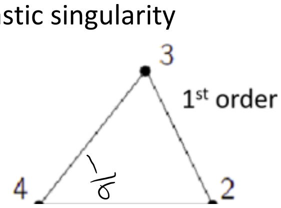


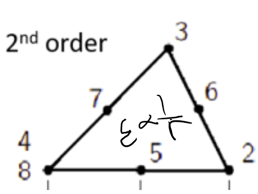


PFM approximating elastic perfectly plastic singularity

- **Elastic-perfectly plastic**

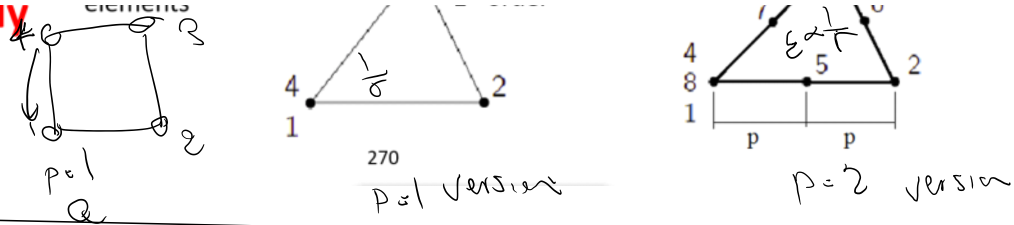




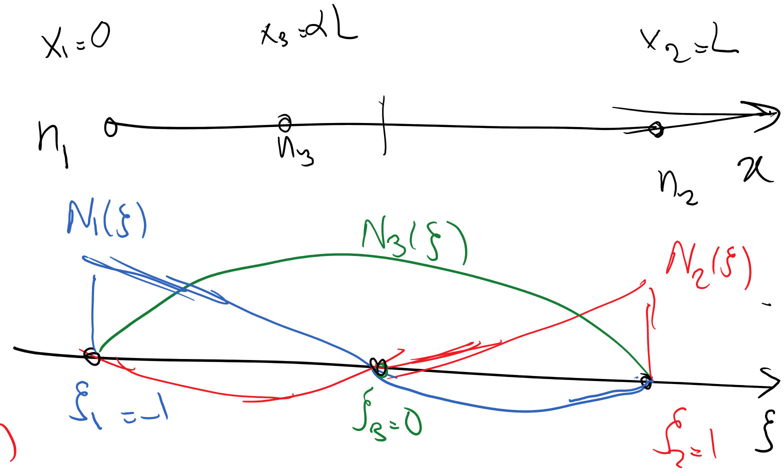


perfectly plastic

$\epsilon : \frac{1}{r}$



Why the singularity is generated?



$$N_1(\xi) = \frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} = \frac{(\xi-1)(\xi-0)}{(-1-1)(-1-0)} = \frac{\xi(\xi-1)}{2}$$

$$N_2(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)} = \frac{(\xi-(-1))(\xi-0)}{(1-(-1))(1-0)} = \frac{\xi(\xi+1)}{2}$$

$$N_3(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)} = \frac{(\xi-(-1))(\xi-1)}{(0-(-1))(0-1)} = 1-\xi^2$$

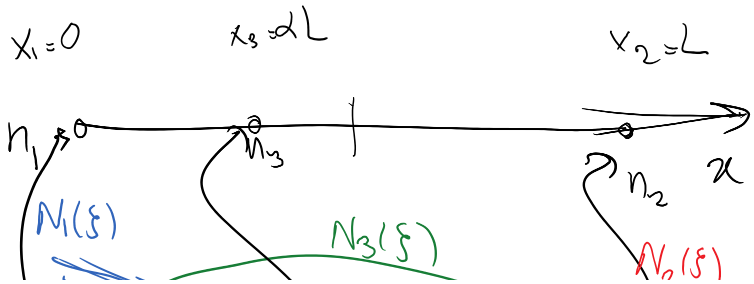
$$\boxed{u(\xi) = N_1(\xi)\,u_1 + N_2(\xi)\,u_2 + N_3(\xi)\,u_3}$$

u_1, u_2, u_3 are nodal displacement values — solution

$$u(\xi=-1) = u_1 N_1(-1) + u_2 N_2(-1) + u_3 N_3(-1) = 1 \cdot u_1$$

Isoparametric

geometry has the same level of approximation as solution

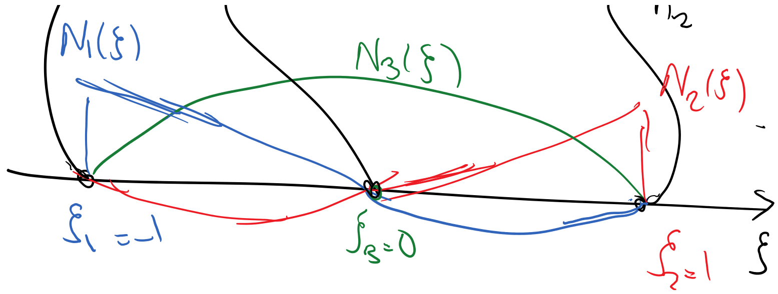


$$x(\xi) = x_1 N_1(\xi) + x_2 N_2(\xi) + x_3 N_3(\xi)$$

obviously $x(\xi_1) = x_1, \ldots$

$x_1 = 0 \quad x_2 = L \quad x_3 = \alpha L$

$$x(\xi) = L\,\frac{\xi(\xi+1)}{2} + (\alpha L)(1-\xi^2)$$



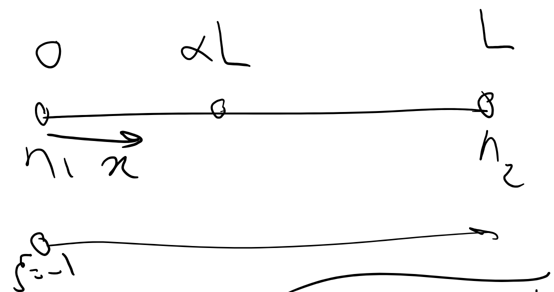
$x_1=0 \quad x_2=L \quad x_3=\alpha L$

$$x(\xi) = L\,\frac{\xi(\xi+1)}{2} + (\alpha L)(1-\xi^2)$$

$$\boxed{x(\xi) = L\left(\alpha + \frac{\xi}{2} + \xi^2\left(\frac{1}{2}-\alpha\right)\right)} \quad (A)$$

Recall $\quad u(\xi) = u_1 N_1(\xi) + u_2 N_2(\xi) + u_3 N_3(\xi)$

I want to achieve the desired $\frac{1}{\sqrt{x}}$ singularity around point ¥ for strain



$$\varepsilon = \frac{du}{dx} = \frac{\frac{du}{d\xi}}{\frac{dx}{d\xi}}$$

$\varepsilon(x=0) \longrightarrow \infty$ requires $\frac{dx}{d\xi} \to 0$ at $\xi=-1$

$$(A) \to \frac{dx}{d\xi} = L\left(\frac{1}{2} + 2\xi\left(\frac{1}{2}-\alpha\right)\right)$$

we want $\frac{dx}{d\xi}(\xi=-1)=0 \longrightarrow \frac{1}{2} - 2\left(\frac{1}{2}-\alpha\right) = 0$

$$\boxed{\alpha = \frac{1}{4}}$$

plug $\alpha = \frac{1}{4}$ in (A)

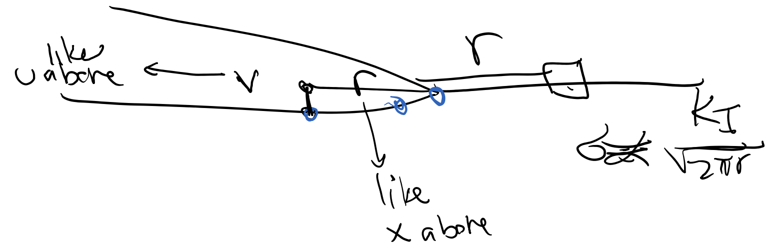
$$\boxed{\text{For } \alpha=\tfrac{1}{4}: \quad x = \frac{L}{4}(\xi+1)^2 \longrightarrow \xi = 2\sqrt{\frac{x}{L}} - 1}$$

$$u = u_1 \underbrace{\frac{\xi(\xi-1)}{2}}_{N_1(\xi)} + u_2 \underbrace{\frac{\xi(\xi+1)}{2}}_{N_2(\xi)} + u_3 \underbrace{(1-\xi^2)}_{N_3(\xi)}$$

$$u = u_1 + \sqrt{\frac{x}{L}}\left(-3u_1 - u_2 + 4u_3\right) + 2\,\frac{x}{L}\left(u_1 + u_2 - 2u_3\right)$$

$$\varepsilon(x) = \frac{du}{dx} = \frac{1}{\sqrt{xL}}\left(-\frac{3}{2}u_1 - \frac{1}{2}u_2 + 2u_3\right) + \frac{2}{L}\left(u_1 + u_2 - 2u_3\right)$$

LEFM



Additional comments about the location of this point

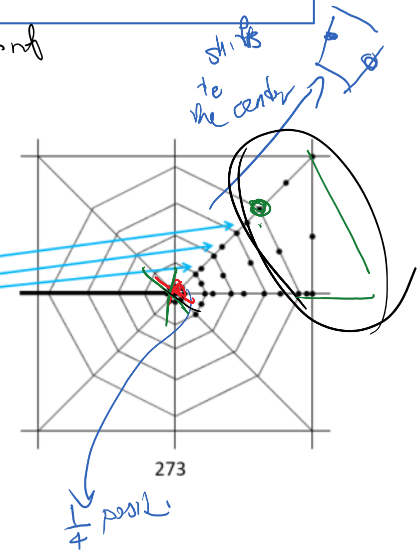
X singularity

$x_1 = 0$ $\quad x_3 = \alpha L$ $\quad x_2 = L$

$\xi_1 = -1$ $\quad \xi_3 = 0$ $\quad \xi_2 = 1$

- Transition elements:
  According to this analysis
  mid nodes of next layers move
  to ½ point from ¼ point

Lynn and Ingraffea 1977)



$$\frac{dx}{d\xi} = L\left(\frac{1}{2} + \xi(1 - 2\alpha)\right)$$

X: singularity at the element edge $\rightarrow \alpha = \frac{1}{4}$

as $\xi \rightarrow -\infty$ for the position singularity

$$\frac{1}{2} + \xi^*(1 - 2\alpha) = 0 \longrightarrow$$

$$\boxed{\alpha = \frac{1}{2} + \frac{1}{4\xi^*}} \qquad \xi^* \rightarrow \infty \qquad \alpha \rightarrow \frac{1}{2}$$

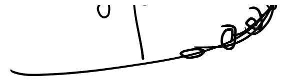
As we'll see, even with interior meshes we can still get decent solutions for K, J, etc as long as the right method is used.

6.1.3 Extraction of K (SIF) and G = J

1 $\quad u_y \quad (v)$

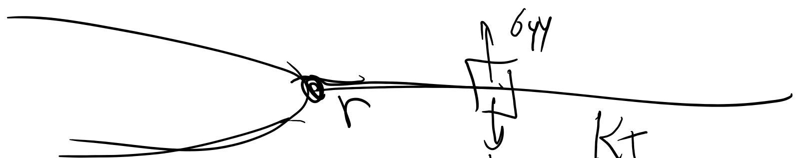
$$2u_y \propto \sqrt{r}\,K \qquad \frac{u_y}{r} \propto K$$

1. u_y (v)



$2u_y \propto \sqrt{r}\,K$ $\quad \frac{u_y}{r} \propto K$

2. σ_{yy}



$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$

$$\boxed{K_I = \sigma_{yy}\sqrt{2\pi r}}$$

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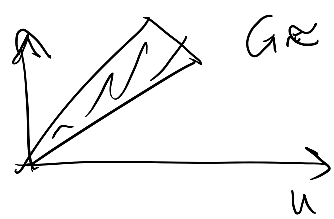
3. K calculated from G (or $\bar{\sigma}$)

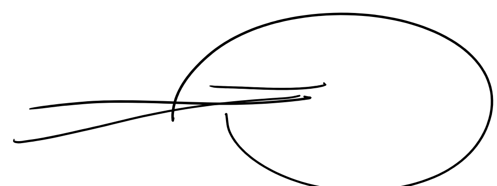
mode I $\quad G = \frac{K_I^2}{E'} \rightarrow K_I = \sqrt{GE'}$



$G \approx \frac{\text{shaded area}}{B\Delta a} = \frac{P^2}{2B}\frac{dC}{da}$ Linear

$G = -\frac{d\Pi}{B\,da}$

$G = J$



→ we'll discuss how these will be used computationally

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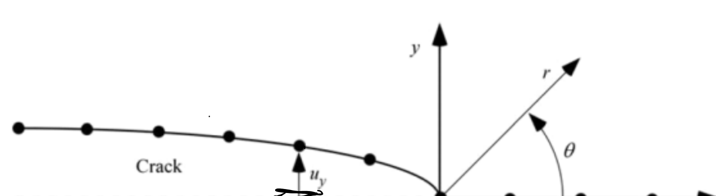
1. K from local fields

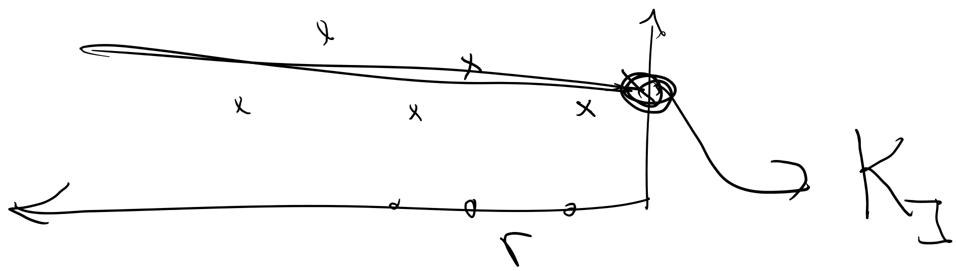
1. Displacement



$K_I(r)$

$$u_y(r,\theta=\pi) = \frac{4K_I\sqrt{r}}{\sqrt{2\pi}E'} \Rightarrow \boxed{K_I = \lim_{r\to 0}\left[\frac{E' u_y}{4}\sqrt{\frac{2\pi}{r}}\right] \quad (\theta=\pi)}$$

$$E' = \begin{cases} E & \text{plane stress} \\ \frac{E}{1-\nu^2} & \text{plane strain} \end{cases}$$

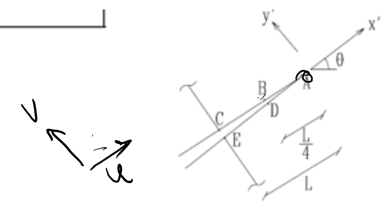


We'll do a best fit to KI(r) and evaluate the function at r = 0

or alternatively from the first quarter point element:

LEFM $v = K_I \dfrac{\kappa+1}{2G}\sqrt{\dfrac{r}{2\pi}}$

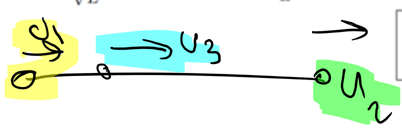
$$u' = \overline{u}'_A + \left(-3\overline{u}'_A + 4\overline{u}'_B - \overline{u}'_C\right)\sqrt{\tfrac{r}{L}} + \left(2\overline{u}'_A + 2\overline{u}'_C - 4\overline{u}'_B\right)\tfrac{r}{L}$$
$$v' = \overline{v}'_A + \left(-3\overline{v}'_A + 4\overline{v}'_B - \overline{v}'_C\right)\sqrt{\tfrac{r}{L}} + \left(2\overline{v}'_A + 2\overline{v}'_C - 4\overline{v}'_B\right)\tfrac{r}{L}$$



$$K_I = \frac{2G}{\kappa+1}\sqrt{\frac{2\pi}{L}}\left(-3\overline{v}'_A + 4\overline{v}'_B - \overline{v}'_C\right)$$

Recall for 1D

$$u = u_1 + \frac{\sqrt{x}}{\sqrt{L}}\left(-3u_1 - u_2 + 4u_3\right) + \frac{2x}{L}\left(u_1 + u_2 - 2u_3\right)$$



$$\begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} = \frac{1}{2}\frac{2G}{\kappa+1}\sqrt{\frac{2\pi}{L}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -3\overline{u}'_A + 4\left(\overline{u}'_B - \overline{u}'_D\right) - \left(\overline{u}'_C - \overline{u}'_E\right) \\ -3\overline{v}'_A + 4\left(\overline{v}'_B - \overline{v}'_D\right) - \left(\overline{v}'_C - \overline{v}'_E\right) \end{bmatrix}$$

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Mixed mode generalization:

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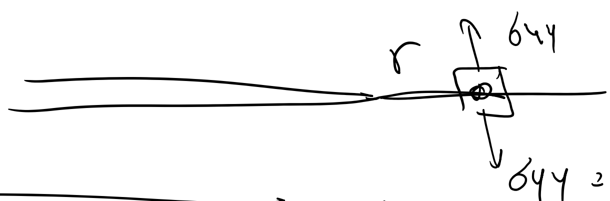
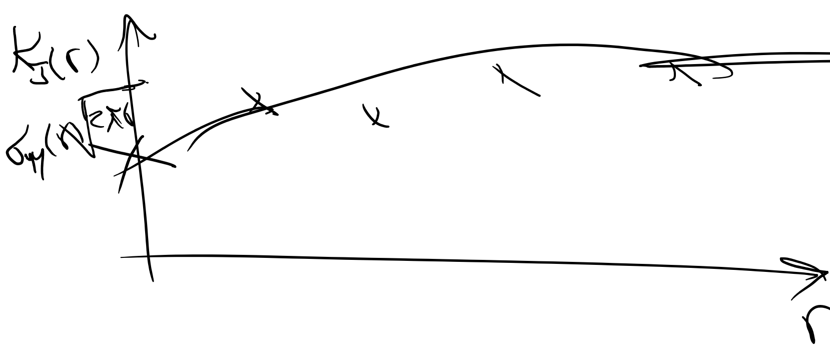
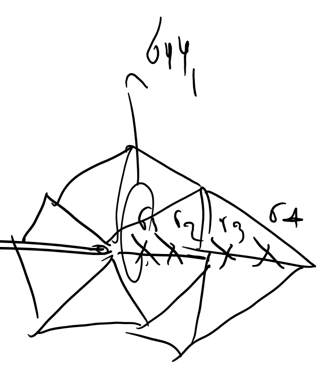
2. Stress:

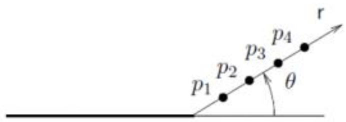


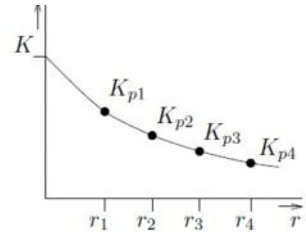
$\sigma_{yy} = \dfrac{K_I}{\sqrt{2\pi r}}$

$$\boxed{K_I = \lim_{r\to 0}\left(\sigma_{yy}\sqrt{2\pi r}\right)}$$









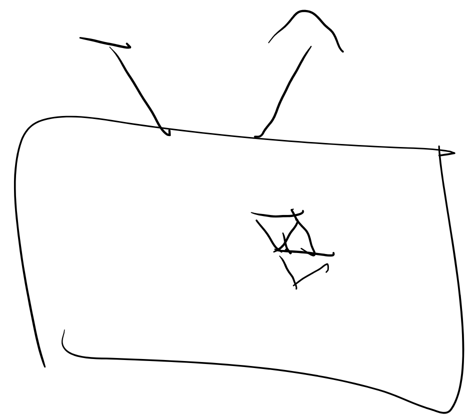
or can be done for arbitrary angle (θ) taking σ angular dependence $f(\theta)$ into account

Reasons why the stress approach is worse:

1.

$$\sigma \propto \frac{1}{\sqrt{r}} \rightarrow \infty \qquad u \propto \sqrt{r} \rightarrow 0$$

Even with quarter point elements & spider web mesh, it's difficult to capture the tendency of the stress field to infinity



2. FEM with smooth element order, enough exact solution

$$\|u^{FEM} - u^{exact}\| = C h^{p+1}$$

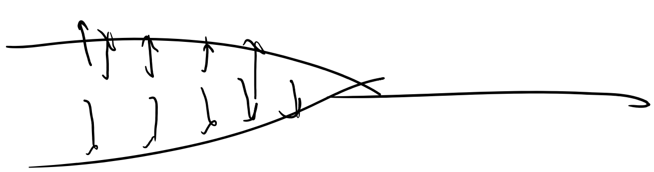
$\varepsilon = \nabla u$ 1 derivative

$\sigma = C \nabla u$

$$\|\sigma^{FEM} - \sigma^{exact}\| = C h^{p}$$

stress solutions are 1 power less accurate

3. Stress approach is specifically poor when the crack surfaces are loaded (pressure vessel, H. Frac)



2. K from energy approaches

1. Elementary crack advance (two FEM solutions for a and $a + \Delta a$)
2. Virtual Crack Extension: Stiffness derivative approach
3. J-integral based approaches (next section)

After obtaining *G* (or *J*=*G* for LEFM) K can be obtained from

$$K_I^2 = E'G \qquad E' = \begin{cases} E & \text{plane stress} \\ \frac{E}{1-\nu^2} & \text{plane strain} \end{cases}$$