2022/10/27

Thursday, October 27, 2022 3:02 PM

2. K from energy approaches

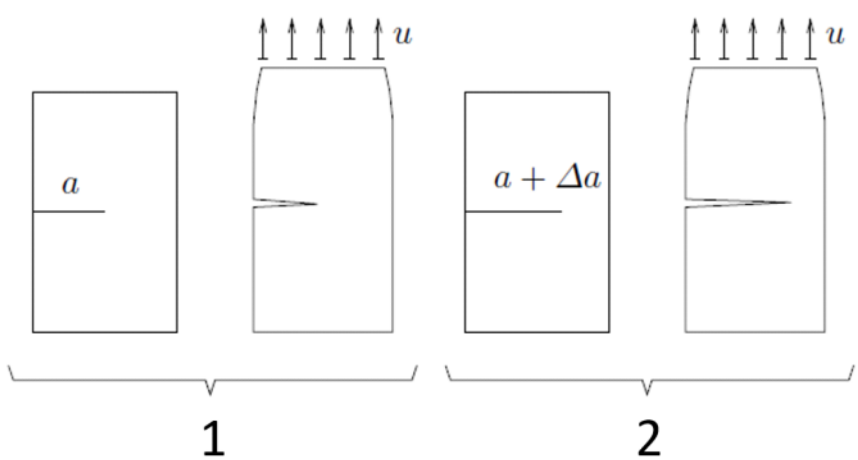
1. Elementary crack advance (two FEM solutions for a and $a + \Delta a$)
2. Virtual Crack Extension: Stiffness derivative approach
3. J-integral based approaches (next section)

After obtaining G (or J=G for LEFM) K can be obtained from

$$K_I^2 = E'G \qquad E' = \begin{cases} E & \text{plane stress} \\ \frac{E}{1-\nu^2} & \text{plane strain} \end{cases}$$

2.1 Elementary crack advance

For fixed grip boundary condition perform two simulations (1, a) and (2, a+Δa):
All FEM packages can compute strain (internal) energy U_i



$$G = -\frac{d\Pi}{dA} = -\frac{d\Pi}{B\,da}$$

We can numerically calculate Π

e.g. FEM $\Pi = \frac{1}{2}u^T K u - u^T P$

u: global displacement vector
P: global force vector
K: stiffness matrix

Finite Difference approximation for this

$G \approx -\left[\Pi(a+\Delta a) - \Pi(a)\right]$

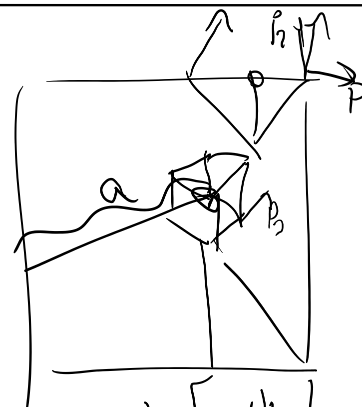
Finite Difference approximation for ...   [K: stiffness matrix

$$G \approx -\left[\frac{\Pi(a+\Delta a) - \Pi(a)}{B \quad \Delta a}\right]$$

Disadvantages : 1. $\begin{cases} \Delta a \to 0 & \text{finite precision \& cancelation error} \\ \Delta a \nearrow & \text{FD approximation of derivative} \end{cases}$

2. Two solutions are needed

2.2 Virtual Crack extension



$$\Pi = \underbrace{U_e}_{\text{internal energy}} - \underbrace{W}_{\text{external work}}$$

$$\Pi = \frac{1}{2} U^t K U - U^t P$$

global stiffness matrix (K)

vector of global unknowns (U)

$$U = \begin{bmatrix} u_1 \\ u_2 \\ | \\ | \\ u_n \end{bmatrix} \quad P = \begin{bmatrix} P_1 \\ | \\ | \\ P_n \end{bmatrix}$$

$$G = \frac{-1}{B}\frac{d\Pi}{da} = \frac{-1}{B}\frac{d}{da}\left(\frac{1}{2}U^t K U - U^t P\right)$$

$$= \frac{-1}{B}\left(\frac{1}{2}\left(\frac{dU}{da}\right)^t K U + \frac{1}{2}U^t\frac{dK}{da}U + \underbrace{\frac{1}{2}U^t K\frac{dU}{da}}_{\text{transpose of this}} - \left(\frac{dU}{da}\right)^t P - U^t\frac{dP}{da}\right)$$

$$= \frac{-1}{B}\left(\frac{1}{2}\left(\frac{dU}{da}\right)^t K U + \frac{1}{2}\left(\frac{dU}{da}\right)^t K^t U + \frac{1}{2}U^t\frac{dK}{da}U - \left(\frac{dU}{da}\right)^t P - U^t\frac{dP}{da}\right)$$

$$= \frac{-1}{B}\left(\left(\frac{dU}{da}\right)^t \underbrace{\left\{\frac{K+K^t}{2}\right\}}_{=K} U + \frac{1}{2}U^t\frac{dK}{da}U - \left(\frac{dU}{da}\right)^t P - U^t\frac{dP}{da}\right)$$

in CFEM K is symmetric

K is symmetric

$$= -\frac{1}{B}\left[\left(\frac{dU}{da}\right)^t \underline{(KU - P)} + \frac{1}{2} U^t \frac{dK}{da} U - U^t \frac{dP}{da}\right]$$

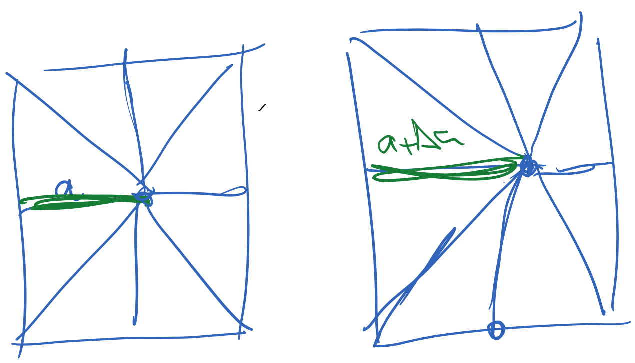
$KU = P \Rightarrow$ provides the solution $\quad KU - P = 0$

$$\boxed{G = \frac{1}{B}\left(-\frac{1}{2} U^t \frac{dK}{da} U + U^t \frac{dP}{da}\right)}$$

this term is often zero as loads do not depend on the crack length

Virtual crack extension

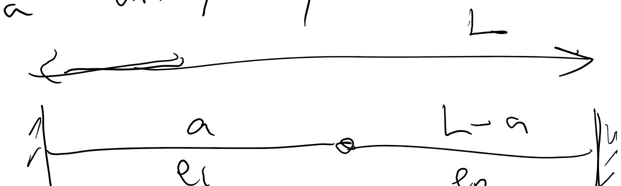
$\frac{dK}{da}$: Approach 1 : Finite Difference



$K(a) \qquad K(a+\Delta a)$

$$\frac{dK}{da} \approx \frac{K(a+\Delta a) - K(a)}{\Delta a}$$

Approach 2: calculate $\frac{dK}{da}$ analytically



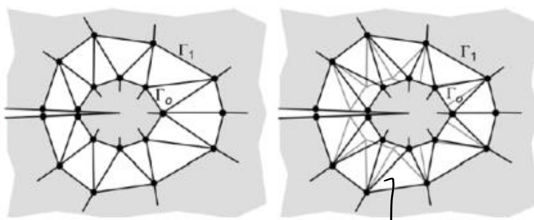
$$K_{e_1} = \frac{(AE)_{e_1}}{a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad K_{e_2} = \frac{(AE)_{e_2}}{L-a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$L-a\,[-1\;\;1]$

$\frac{dK_{e1}}{da} = \ldots \qquad \frac{dK_{e2}}{da} = \;''$

assemble these as done in FEM to get $\frac{dK}{da}$

or
or
of



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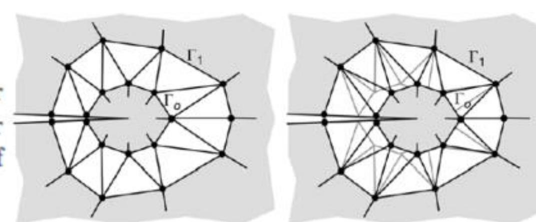
analytical $\frac{dK}{da}$ of elements $\Longrightarrow$ $\underbrace{\frac{1}{2} U^t \frac{dK}{da} U}_{G}$

1. No further approximation beyond FEM solution
2. Only 1 K evaluation & 1 $U = K^{-1}P$ solution

- Only the few elements that are distrorted contribute to $\frac{\partial K}{\partial a}$
- We may not even need to form elements and assemble K for a and $a + \Delta a$ to obtain $\frac{\partial K}{\partial a}$. We can explicitly obtain $\frac{\partial k^e}{\partial a}$ for elements affected by crack growth by computing derivatives of actual geometry of the element to parent geometry.



- This method is equivalent to J integral method (Park 1974)

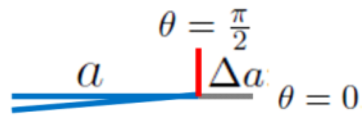
2.2 Virtual crack extension: Mixed mode

- For LEFM energy release rates G_1 and G_2 are given by

$$\begin{aligned} J_1 &= G_1 = \frac{K_I^2 + K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \\ J_2 &= G_2 = \frac{-2K_I K_{II}}{E'} \end{aligned}$$

- Using Virtual crack extension (or elementary crack advance) compute G_1 and G_2 for crack lengths a, $a + \Delta a$

$$\begin{aligned} J_1 &= G_1 = \frac{K_I^2 + K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \\ J_2 &= G_2 = \frac{-2K_I K_{II}}{E'} \end{aligned}$$



- Obtain K_I and K_{II} from:

$$\boxed{\begin{aligned} K_I &= \frac{s \pm \sqrt{s^2 + \frac{8G_2}{\alpha}}}{4} \\ K_{II} &= \frac{s \mp \sqrt{s^2 + \frac{8G_2}{\alpha}}}{4} \end{aligned}}$$

Note that there are two sets of solutions!

$$s = 2\sqrt{\frac{G_1 - G_2}{\alpha}} \text{ and } \alpha = \frac{(1+\nu)(1+\kappa)}{E}$$

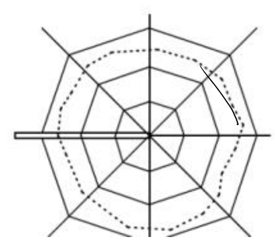
$G = -\frac{d\Pi}{B\,da}$

$G = J$ integral, we can numerically calculate the J integral

Methods to evaluate J integral:

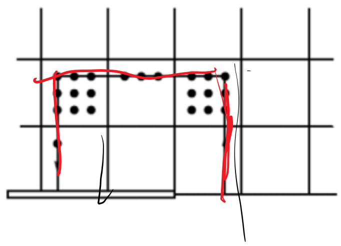
1. Contour integral:

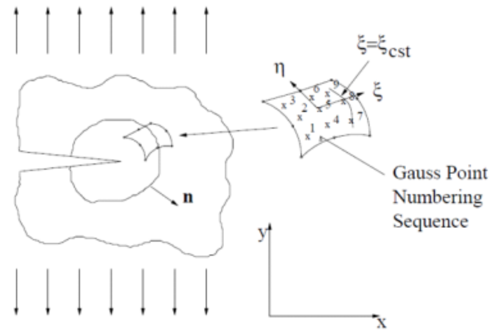
$$J_1 = \int_\Gamma \left(w\,dy - \mathbf{t}\frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)$$
$$J_2 = \int_\Gamma \left(-w\,dx - \mathbf{t}\frac{\partial \mathbf{u}}{\partial y} d\Gamma \right)$$



J integral: 1.Contour integral

- Stresses are available and also more accurate at Gauss points
- Integral path goes through Gauss points





$$J = \int_\Gamma w dy - \mathbf{t} \cdot \frac{\partial \mathbf{d}}{\partial x} ds \Rightarrow J^e = \int_{-1}^{1} \left\{ \underbrace{\frac{1}{2}\left[\sigma_x \frac{\partial u}{\partial x} + \tau_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \sigma_y \frac{\partial v}{\partial y}\right]}_{w} \underbrace{\frac{\partial y}{\partial \eta}}_{dy} - \underbrace{\left[(\sigma_x n_1 + \tau_{xy} n_2)\frac{\partial u}{\partial x} + (\tau_{xy} n_1 + \sigma_y n_2)\frac{\partial v}{\partial x}\right]}_{\mathbf{t}\cdot\frac{\partial \mathbf{d}}{\partial x}} \underbrace{\sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2}}_{ds} \right\} d\eta = \int_{-1}^{1} I d\eta$$

Cumbersome to formulate the integrand, evaluate normal vector, and integra over lines (2D) and surfaces (3D)

Not commonly used

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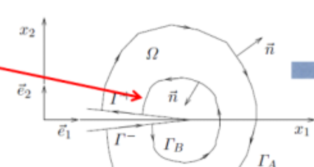
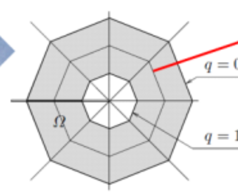
2. Equlvalent (Energy) **domain** integral (**EDI**):
   - Gauss theorem: line/surface (2D/3D) integral surface/volume integral
   - Much simpler to evaluate computationally
   - Easy to incorporate plasticity, crack surface tractions, thermal strains, *etc.*
   - Prevalent method for computing J-integral

Line integral





Volume integral

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Energy release rate calculated from J integral:

1. No body force
2. No inertia effect, (acceleration is zero)
3. Nonlinear elasticity and no plastic unloading
4. Homogeneity
5. No thermal strain
6. Traction free crack surfaces

A new version of the J integral addresses all these issues:

General form of J integral

$$w = \int^{\varepsilon_{kl}^m} \sigma_{ij} d\varepsilon_{ij}^m$$

General form of J integral

$$w = \int_0^{\varepsilon_{kl}^m} \sigma_{ij} d\varepsilon_{ij}^m$$

$$J = \lim_{\Gamma_o \to 0} \int_{\Gamma_o} \left[(w + T)\delta_{1i} - \sigma_{ij} \frac{\partial u_j}{\partial x_1} \right] n_i d\Gamma$$

Inelastic stress

addresses #2

Kinetic energy density

$$T = \frac{1}{2}\rho \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t}$$

Can include (visco-) plasticity, and thermal stresses

$$\varepsilon_{ij}^{total} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p} + \alpha\Theta\delta_{ij} = \varepsilon_{ij}^{m} + \varepsilon_{kk}^{t}$$

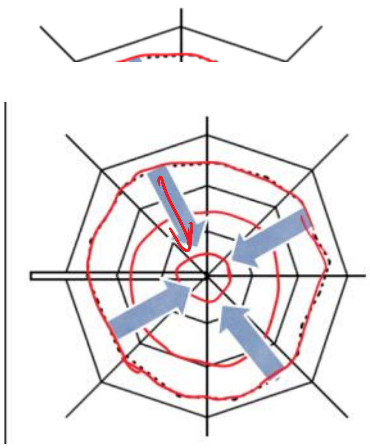
Elastic

Plastic

Thermal (Θ temperature)

addresses #5

addresses #3



we need $\Gamma_o \to 0$ to add address #2,3,5

$\Gamma_o \to 0$: J contour approaches Crack tip (CT) ⇒

Accuracy of the solution deteriorates at CT ⇒

Inaccurate/Impractical evaluation of J using contour integral as $\Gamma_o \to 0$ because numerical results are very bad around the crack tip

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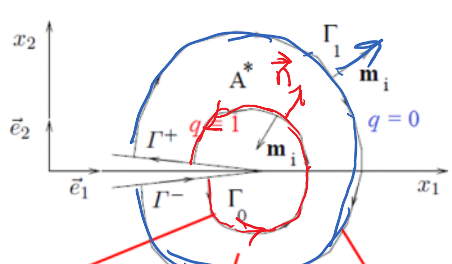
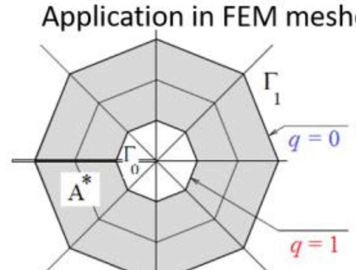
Divergence theorem: Line/Surface (2D/3D) integ



Original J integral contour

Surface integral after using divergence theorem

Application in FEM meshes

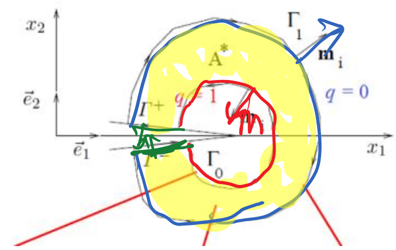


$\Gamma_o \to 0$ ⇒ 2D mesh covers crack tip

Surface integral after using divergence theorem

$$J = \int_{\Gamma_0} \left[(w+T)\delta_{i1} - \sigma_{ij}\frac{\partial u_i}{\partial x_1} \right] n_1 \, q \, d\Gamma$$

→ I can add this as on Γ_0 we choose the ~~funct~~ $q = 1$



$$-J = \int_{\Gamma_0} \left[(w+T)\delta_{i1} - \sigma_{ij}\frac{\partial u_i}{\partial x_1} \right] m_i q \, d\Gamma$$

$$+ \int \left[(w+T)\delta_{i1} - \sigma_{ij}\frac{\partial u_i}{\partial x_1} \right] m_i q \, d\Gamma$$

this is zero

$$+ \int_{\Gamma^- \cup \Gamma^+} \left[\underbrace{(w+T)\delta_{i1}}_{0} - \underbrace{\sigma_{ij}\frac{\partial u_i}{\partial x_1}}_{} \right] m_i q \, d\Gamma + \int_{\Gamma_1 \cup \Gamma_2} \sigma_{ij}\frac{\partial u_i}{\partial x_1} m_i q \, d\Gamma$$

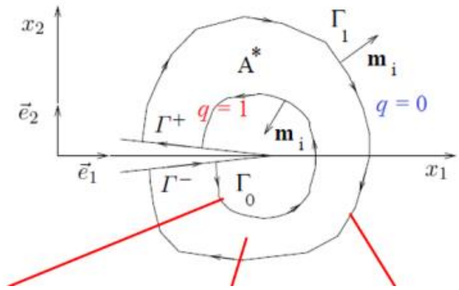
$\delta_{i1} m_i \, d\Gamma = dx_2$

$$-J = \int_{\partial A^*} \left[(w+T)\delta_{i1} - \sigma_{ij}\frac{\partial u_i}{\partial x_1} \right] m_i q \, d\Gamma + \int_{\Gamma_1 \cup \Gamma_2} \sigma_{ij}\frac{\partial u_i}{\partial x_1} m_i q \, d\Gamma$$

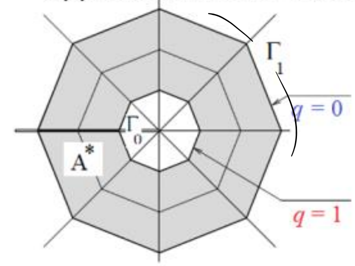
apply the Gauss theorem

$$-J = \int_{A^*} \frac{\partial}{\partial x_i} \left\{ \left[(w+T)\delta_{i1} - \sigma_{ij}\frac{\partial u_i}{\partial x_1} \right] q \right\} dA + \int \sigma_{ij}\frac{\partial u_i}{\partial x_1} m_i q \, d\Gamma$$

take the limit $\Gamma \to 0$

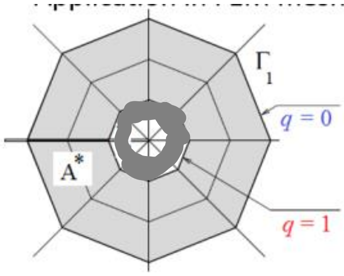


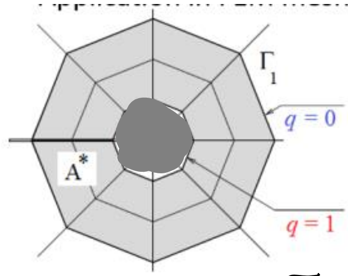
Application in FEM meshes



$\Gamma_0 \rightarrow 0$ 2D mesh covers crack tip

we cover the whole area!

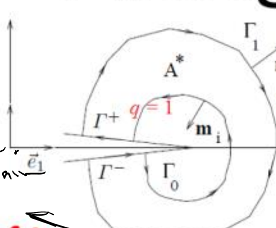


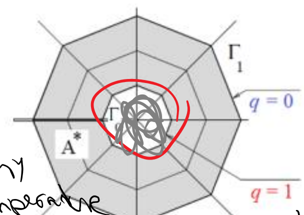


limit of $\Gamma_0 \rightarrow 0$ covers the whole area

$$\boxed{J = \int_{A^*} \frac{\partial}{\partial x_i}\left\{\left[\sigma_{ij}\frac{\partial u_j}{\partial x_1} - w\delta_{1i}\right]q\right\}dA - \int_{\Gamma_+ + \Gamma_-}\sigma_{2j}\frac{\partial u_j}{\partial x_1}q\,d\Gamma}$$

J integral: 2. EDI





plastic strain

plastic energy

temperature

bodyforce

General form of J

$$J = \int_{A^*}\left\{\left[\sigma_{ij}\frac{\partial u_j}{\partial x_1} - w\delta_{1i}\right]\frac{\partial q}{\partial x_i} + \left[\sigma_{ij}\frac{\partial \varepsilon_{ij}^p}{\partial x_1} - \frac{\partial w^p}{\partial x_1} + \alpha\sigma_{ii}\frac{\partial \Theta}{\partial x_1} - F_i\frac{\partial u_j}{\partial x_1}\right]q\right\}dA - \int_{\Gamma_+ + \Gamma_-}\sigma_{2j}\frac{\partial u_j}{\partial x_1}q\,d\Gamma$$

Plasticity effects

Thermal effects

Body force

Nonzero crack surface traction

often these terms are zero

Simplified case:
Nonlinear (elastic), no thermal strain, no body force (F), and traction free surfaces

Simplified Case:

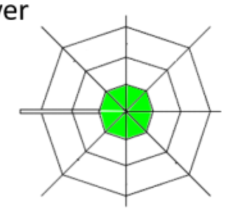
(Nonlinear) elastic, no thermal strain, no body force, traction free crack surfaces

$$J = \int_{A^*}\left[\sigma_{ij}\frac{\partial u_j}{\partial x_1} - w\delta_{1i}\right]\frac{\partial q}{\partial x_i}\,dA$$

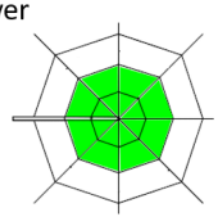
= 0 in first layer around the crack / we don't need to worry about bad solution there

J integral: 2. EDI FEM Aspects

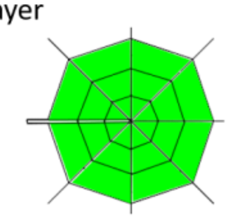
- Since $J_0 \to 0$ the inner J_0 collapses to the crack tip (CT)
- J_1 will be formed by element edges
- By using spider web (rozet) meshes any <u>reasonable</u> number of layers can be used to compute J:



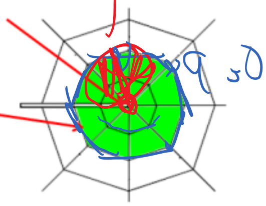
1 layer



2 layer

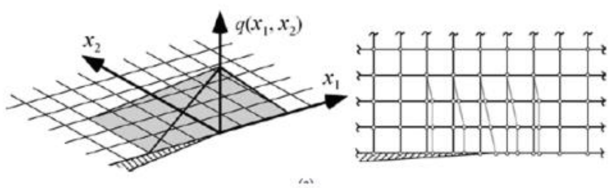


3 layer

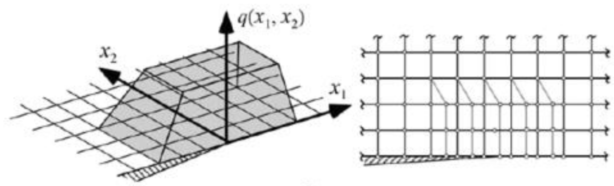


J integral: 2. EDI FEM Aspects

- Shape of decreasing function ***q***:

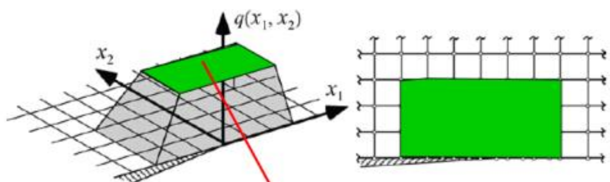


Pyramid *q* function



Plateau *q* function

- Plateau q function useful when inner elements are not very accurate: e.g. when singular/quarter point elements are not used



$$\frac{\partial q}{\partial x_i} = 0$$ These elements do not contribute to J

$$J = \int_{A^*} \left[\sigma_{ij} \frac{\partial u_j}{\partial x_1} - w\delta_{1i} \right] \frac{\partial q}{\partial x_i} dA$$