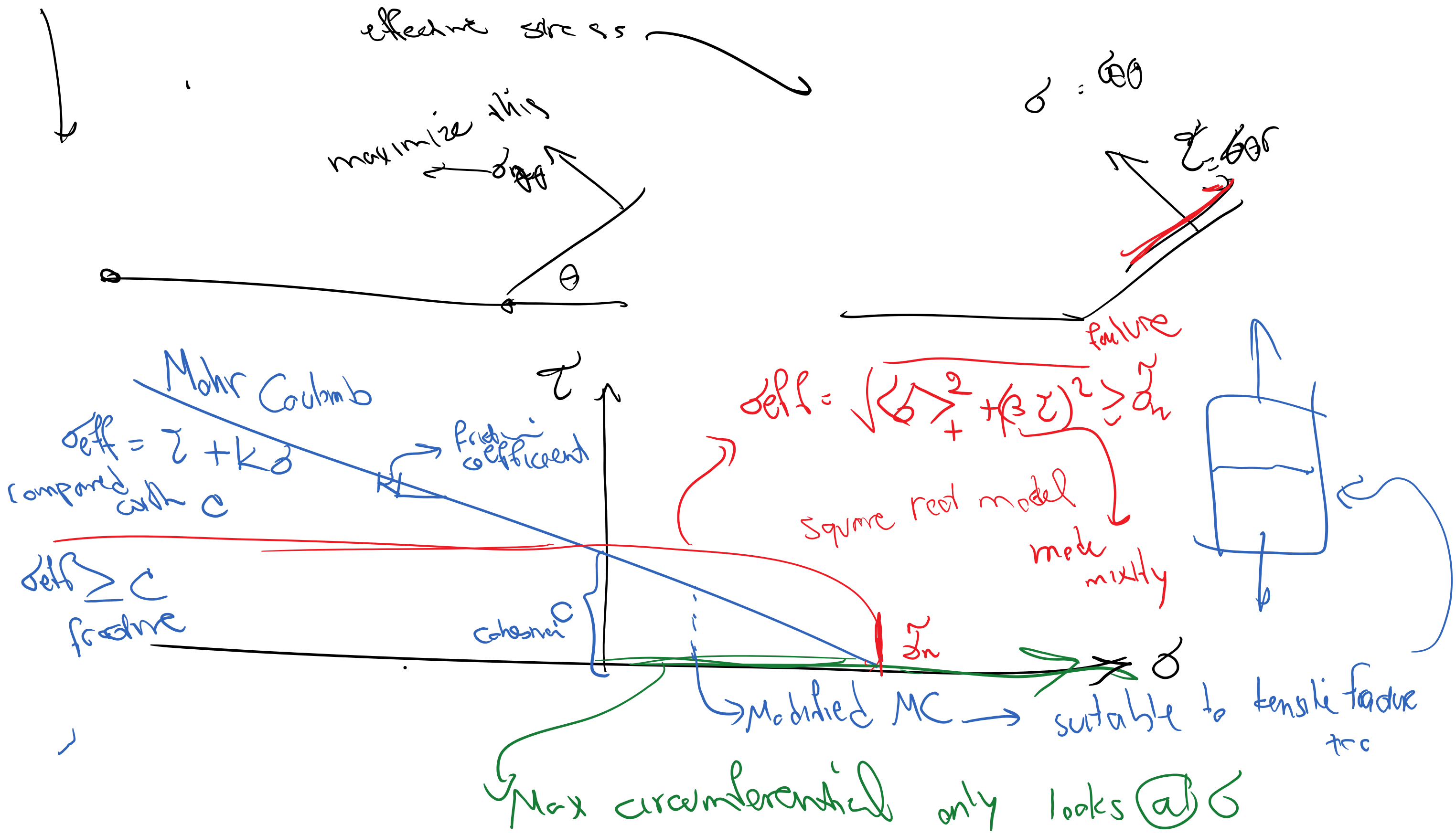
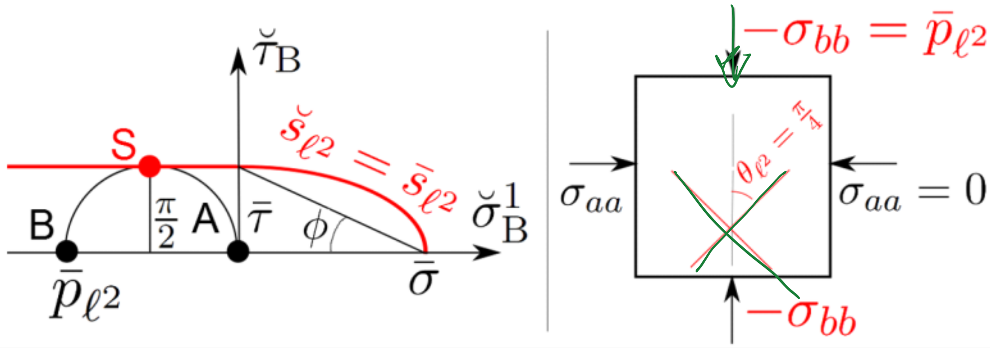
2022/11/15

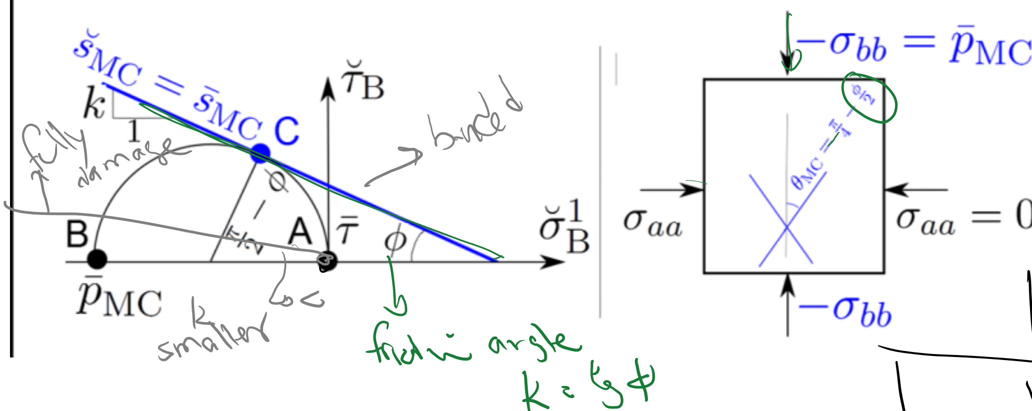
Tuesday, November 15, 2022 2:27 PM

Generalization of max circumferential stress

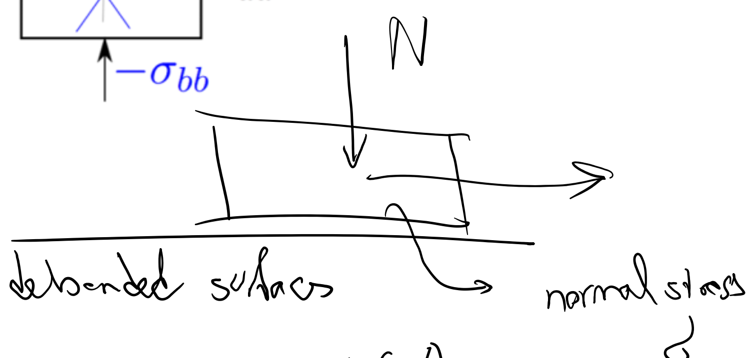


Compressive strength: Comparison of the two models





Steeper angle and higher strength for MC model



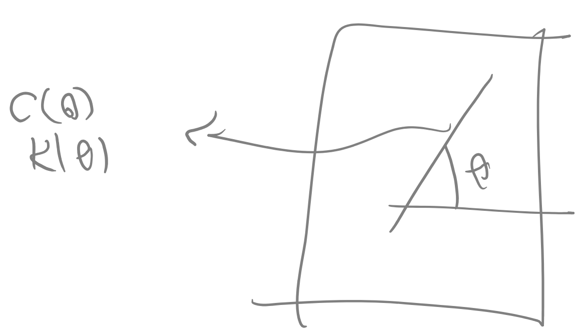
$$\tau = -\underline{k}\,\sigma + c$$

$\boxed{\text{if } c = 0}$

$$\tau = k(-\sigma)$$

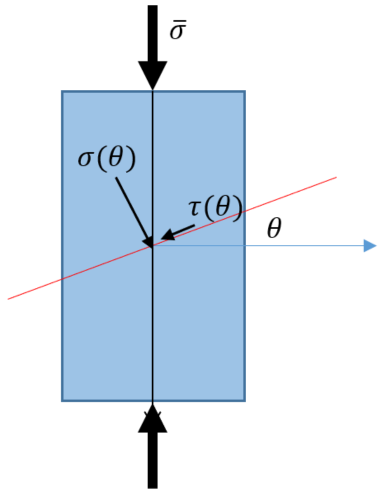
Modeling anisotropy

Failure criterion properties are orientation-dependent

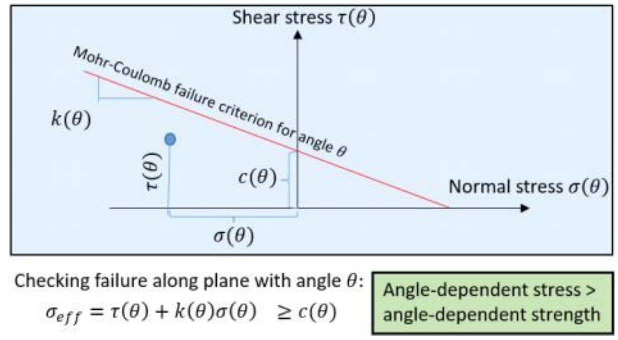


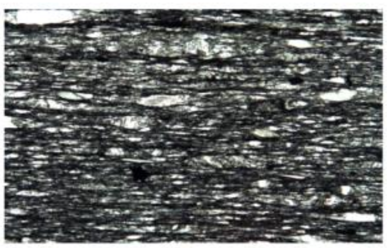
2.1 Definitions

Resolved normal and shear stresses, $\sigma(\theta), \tau(\theta)$



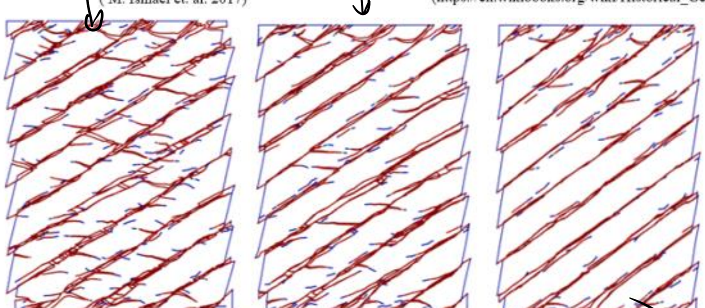
Definitions of loading direction, failure plane, normal and shear stresses



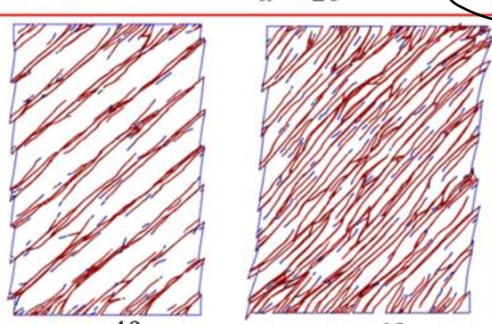


anisotropy in a metamorphic slate.
(M. Ismael et. al. 2017)

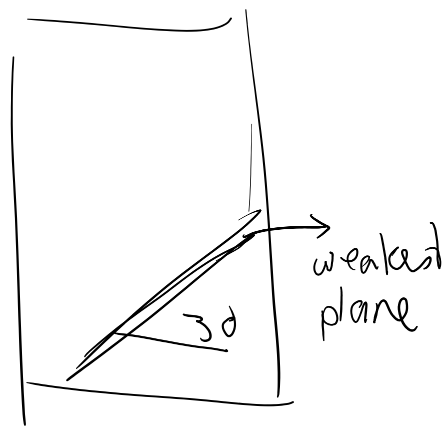
Sandstone, The Wave, Arizona
(https://en.wikibooks.org/wiki/Historical_Geology/Sedimentary_rocks)



$\alpha = 10$ $\alpha = 20$ $\alpha = 30$

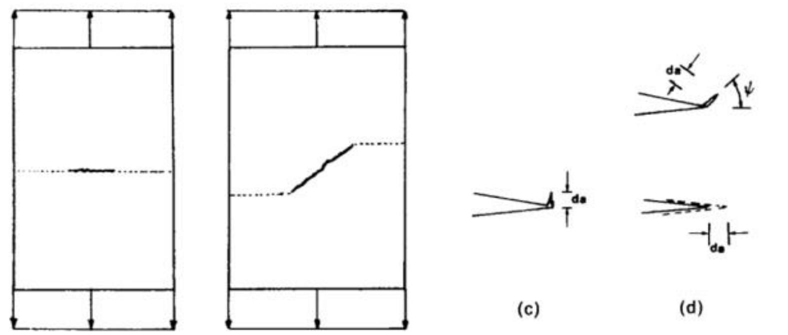


$\alpha = 40$ $\alpha = 60$



2nd Propagation criterion:

Maximum Energy Release Rate

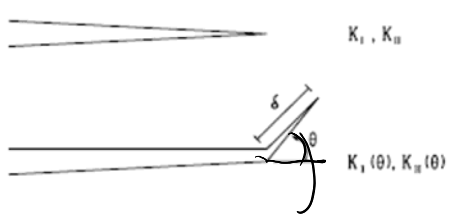


G: crack driving force -> crack will grow in the direction that G is maximum

Erdogan, F. and Sih, G.C. 1963

"If we accept Griffith (energy) theory as the valid criteria which explains crack growth, then the crack will grow in the direction along which the elastic energy release per unit crack extension will be maximum and the crack will start to grow when this energy reaches a critical value (or $G = G(\delta, \theta)$). Evaluation of $G(\delta, \theta)$ poses insurmountable mathematical difficulties."

Maximum Energy Release Rate



Look for all θ & calculate $G(\theta)$

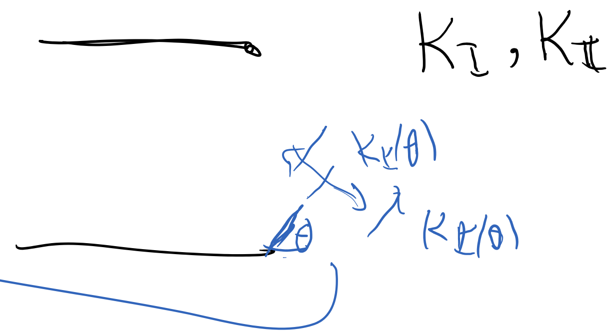
if $G(\theta) = R(\theta)$ the crack propagates at θ_c

Isotropic $\theta_0 \ni$ $G(\theta_0) \geq G(\theta)$

$G(\theta_0) = R$ or not

$$\left\{ \begin{array}{c} K_I(\theta) \\ K_{II}(\theta) \end{array} \right\} = \left(\frac{4}{3 + \cos^2\theta} \right) \left(\frac{1 - \frac{\theta}{\pi}}{1 + \frac{\theta}{\pi}} \right)^{\frac{\theta}{2\pi}} \left\{ \begin{array}{c} K_I \cos\theta + \frac{3}{2} K_{II} \sin\theta \\ K_{II} \cos\theta - \frac{1}{2} K_I \sin\theta \end{array} \right\}$$

$$G(\theta) = \frac{1}{E'}\left(K_I^2(\theta) + K_{II}^2(\theta)\right)$$

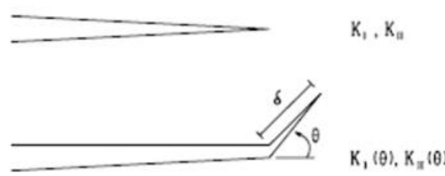


$$G(\theta) = \frac{4}{E'}\left(\frac{1}{3+\cos^2\theta}\right)^2\left(\frac{1-\frac{\theta}{\pi}}{1+\frac{\theta}{\pi}}\right)^{\frac{\theta}{\pi}}$$
$$[(1+3\cos^2\theta)K_I^2 + 8\sin\theta\cos\theta K_I K_{II} + (9-5\cos^2\theta)K_{II}^2]$$

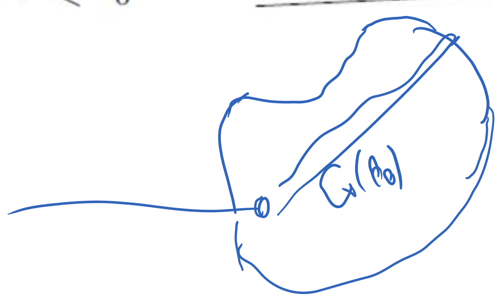
Find the angle for which G is maximum

Maximization condition

$$\frac{\partial G(\theta)}{\partial\theta} = 0$$
$$\frac{\partial^2 G(\theta)}{\partial\theta^2} < 0$$



θ_0



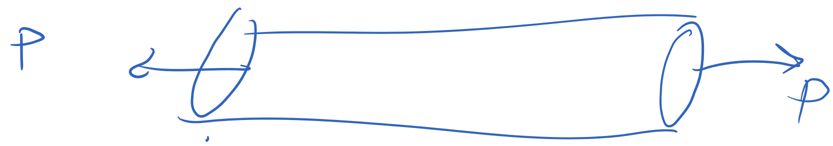
$G(\theta) = R$ or not

$$G(\theta) = \frac{4}{E'}\left(\frac{1}{3+\cos^2\theta}\right)^2\left(\frac{1-\frac{\theta}{\pi}}{1+\frac{\theta}{\pi}}\right)^{\frac{\theta}{\pi}}$$
$$[(1+3\cos^2\theta)K_I^2 + 8\sin\theta\cos\theta K_I K_{II} + (9-5\cos^2\theta)K_{II}^2]$$

$$\boxed{4\left(\frac{1}{3+\cos^2\theta_0}\right)^2\left(\frac{1-\frac{\theta_0}{\pi}}{1+\frac{\theta_0}{\pi}}\right)^{\frac{\theta_0}{\pi}} \left[\left(1+3\cos^2\theta_0\right)\left(\frac{K_{\mathrm{I}}}{K_{\mathrm{Ic}}}\right)^2 + 8\sin\theta_0\cos\theta_0\left(\frac{K_{\mathrm{I}}K_{\mathrm{II}}}{K_{\mathrm{Ic}}^2}\right) + \left(9-5\cos^2\theta_0\right)\left(\frac{K_{\mathrm{II}}}{K_{\mathrm{Ic}}}\right)^2\right] = 1}$$

Strain Energy Density (SED) criterion

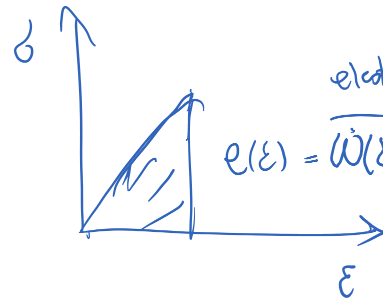
Sih 1973



strain energy density

$$\sigma = \frac{\partial \omega(\varepsilon)}{\partial \varepsilon}$$

$$\omega(\varepsilon) = \sigma \, d\varepsilon$$



elastic elastic

$$e(\varepsilon) = \omega(\varepsilon) = \frac{1}{2}\sigma\varepsilon = \frac{E}{2}\varepsilon^2$$

linear 1D

$$U = \int_{\Omega} \omega(\varepsilon)\, dV = \int_{\Omega} \sigma : d\varepsilon$$

strain energy

$$W(\theta) = \frac{1}{4\mu}\left[\frac{\kappa+1}{4}(\sigma_x^2+\sigma_y^2) - 2(\sigma_x\sigma_y - \tau_{xy}^2)\right]$$

LEFM we have all asymptotic stress soln

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \Sigma_x^{I}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} \Sigma_x^{II}(\theta)$$

$$\sigma_y = \qquad \Sigma_y^{I} \qquad \Sigma_y^{II}$$

$$\tau_{xy} =$$

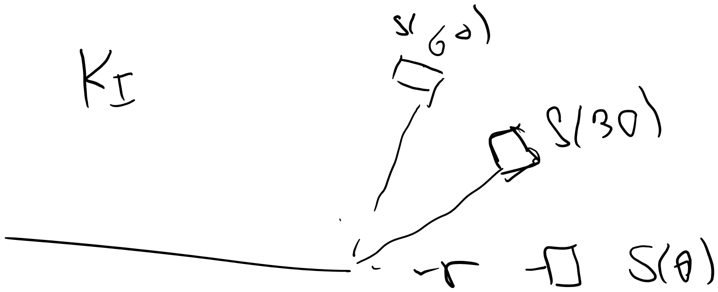
$$\sigma_x = \frac{K_\mathrm{I}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) - \frac{K_\mathrm{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$\sigma_y = \frac{K_\mathrm{I}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + \frac{K_\mathrm{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \quad (7.13)$$

$$\tau_{xy} = \frac{K_\mathrm{I}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} + \frac{K_\mathrm{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right).$$

$$\omega = \frac{S(\theta)}{r}$$

K_I

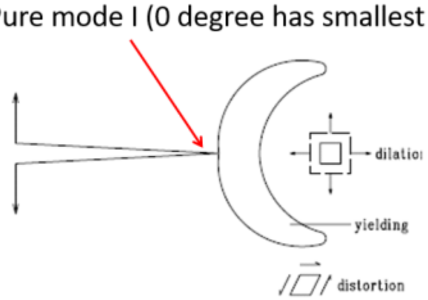


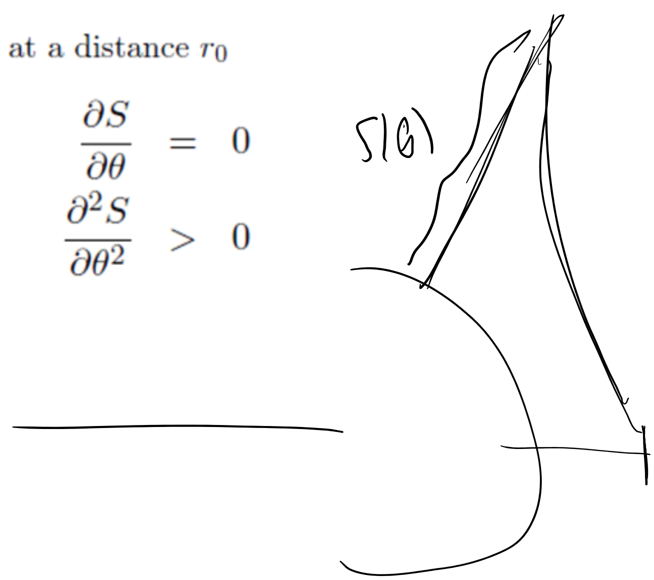
- Crack direction θ_0 which **minimizes** the strain energy density S
- Crack Extends when S reaches a critical value at a distance r_0

Minimization condition

$$\frac{\partial S}{\partial \theta} = 0$$
$$\frac{\partial^2 S}{\partial \theta^2} > 0$$

Pure mode I (0 degree has smallest S)





After finding the angle from minimizing S, we can check if a crack propagates or not

Strain Energy Density (SED) criterion

$$\boxed{\frac{8\mu}{(\kappa-1)}\left[a_{11}\left(\frac{K_I}{K_{Ic}}\right)^2+2a_{12}\left(\frac{K_I K_{II}}{K_{Ic}^2}\right)+a_{22}\left(\frac{K_{II}}{K_{Ic}}\right)^2\right]\geq 1}$$

$$a_{11} = \frac{1}{16\mu}\left[(1+\cos\theta)(\kappa-\cos\theta)\right]$$

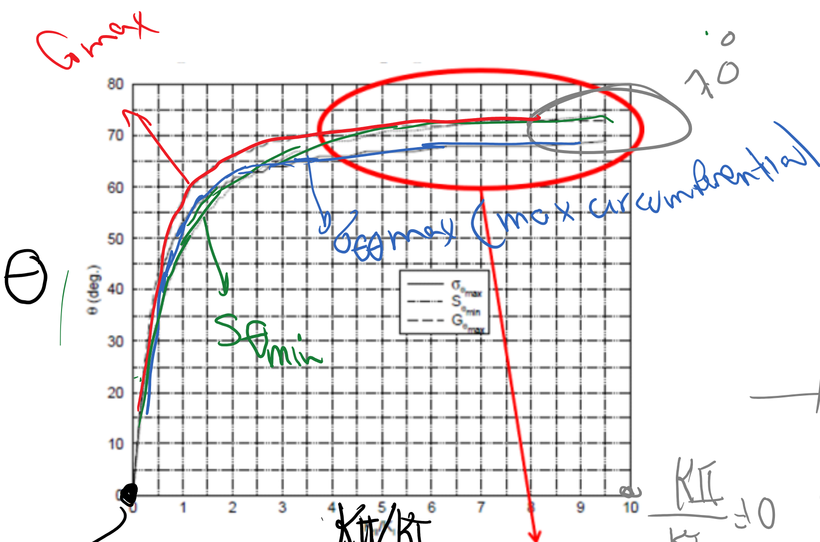
$$a_{12} = \frac{\sin\theta}{16\mu}\left[2\cos\theta-(\kappa-1)\right]$$

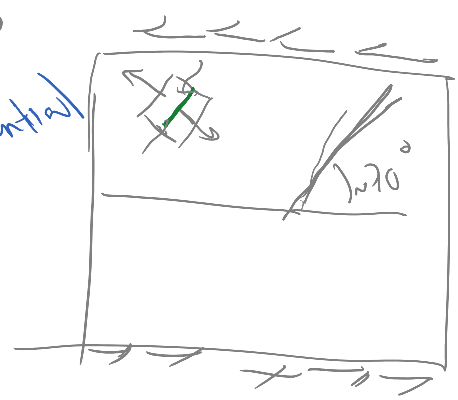
$$a_{22} = \frac{1}{16\mu}\left[(\kappa+1)(1-\cos\theta)+(1+\cos\theta)(3\cos\theta-1)\right]$$

→ optimal angle that minimizes θ

Poisson ratio dependent

$$\begin{cases}\kappa = \frac{3-\nu}{1+\nu} & \text{(plane stress)}\\ \kappa = 3-4\nu & \text{(plane strain)}\end{cases}$$



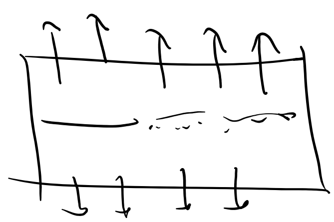


$\frac{K_{II}}{K_I} \approx 10$ $K_{II} >> K_I$ almost pure mode 2

~ 70 degree angle for mode II !

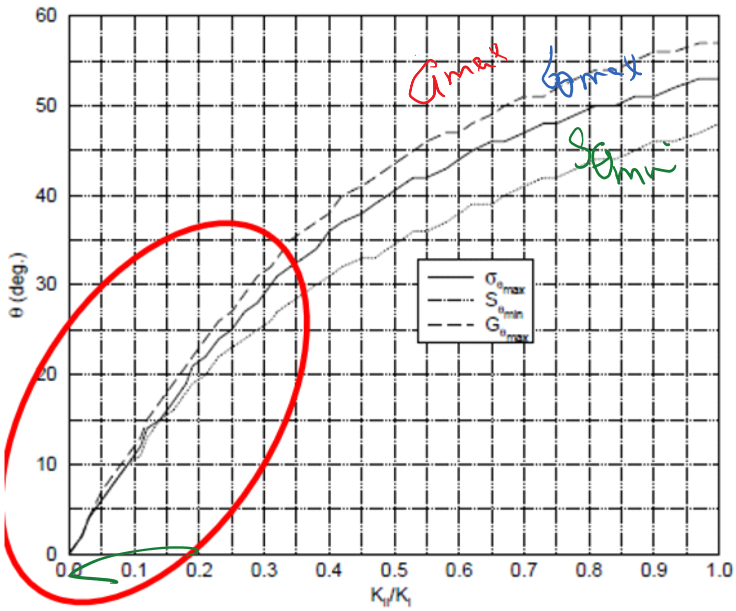
$K_{II} = 1 \longrightarrow \theta = 0$

pure mode I



For pure mode II or very mode II dominant loading the crack propagates by theta ~ 70 degrees and aligns itself to a state that is much closer to mode I. (the crack always extend in a direction that ties to "minimize" KII/KI

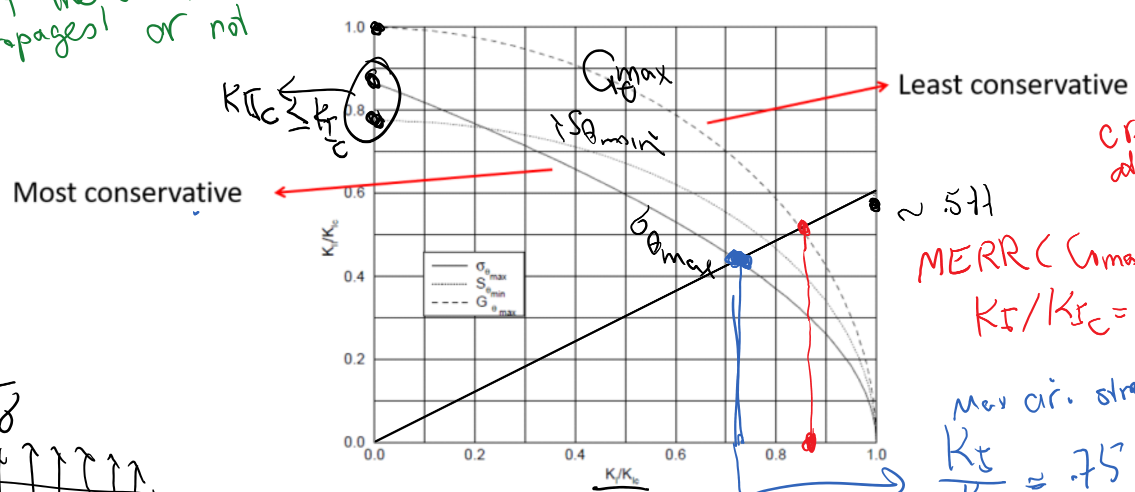
How about for close to mode I



Zoom view (low K_{II} component)

For low KII values, all criteria give more or less the same propagation direction -> overall they all provide similar crack paths.

if the crack propages, or not



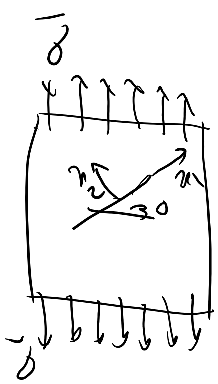
crack propagates at the highest load

MERR (G_{max})

$K_I/K_{Ic} = .835$

Max cir. stress

$\frac{K_I}{K_{Ic}} \approx .75$ crack propagates at the smallest load



$$K_I = \sqrt{\pi a}\,\bar{\sigma}\cos^2\theta$$

$$K_{II} = \sqrt{\pi a}\,\bar{\sigma}\sin\theta\cos\theta$$

$$\theta = 30^\circ$$

$$\rightarrow \frac{K_{II}}{K_I} = \tan\theta = \frac{1}{\sqrt{3}} = 0.577$$

Crack nucleation:

Crack nucleation criterion should be consistent with the crack propagation criterion.

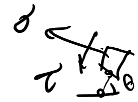
LEFM ... $\sigma_{eff}(\sigma,\tau) > \bar{\sigma}(\theta)$ ...

Crack nucleation criterion should be consistent with the crack propagation criterion.

LEFM

Difficult

$\sigma_{eff}(\sigma,\tau) > \tilde{\sigma}(\theta)$ strength as a function of angle



$\sigma_{eff}(\theta)$

compare with $\tilde{\sigma}(\theta)$

MERR

if $G(\theta) = R$ and max over θ crack propagates

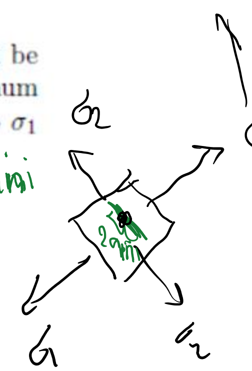
Crack nucleation criterion

1st principle

- For Maximum Energy Release Rate Criterion if we assume there are no defects, there will be no crack nucleation. However, assuming that local stress field generates a tensile maximum principal stress of σ_1 a "microscopic" initial crack (defect) of length a_{ini} perpendicular to σ_1 direction generates,

$$G = \frac{K_I^2 + K_{II}^2}{E'} = \pi a_{\text{ini}} \sigma_1^2$$

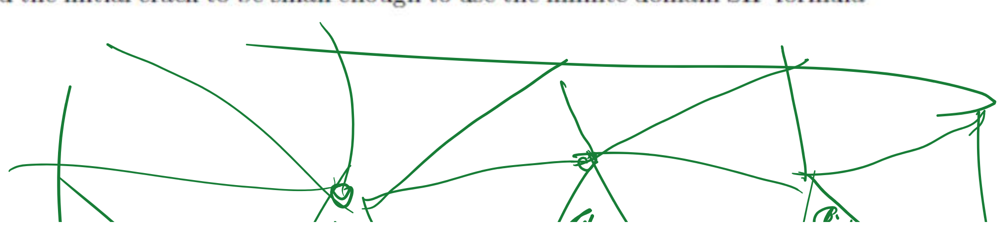
$K_I = \sigma_1\sqrt{\pi a_{ini}}$
$K_{II} = 0$

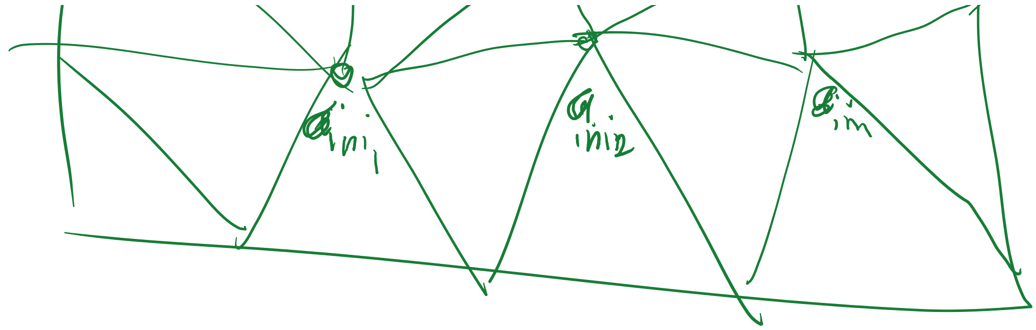


so the microcrack propagates (*i.e.,* a "macroscopic" crack nucleates) when,

$$G = G_c \quad \Leftrightarrow \quad \sigma_1 = \sqrt{\frac{G_c}{\pi a_{\text{ini}}}}$$

- Initial crack direction perpendicular to σ_1 is chosen to maximize G.
- We have assumed the initial crack to be small enough to use the infinite domain SIF formula of $K_I = \sqrt{\pi a}\bar{\sigma}$.

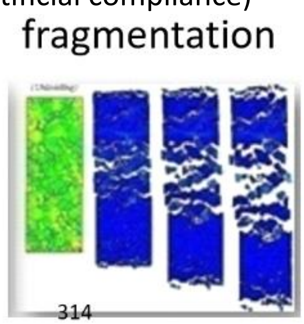




Choices for aini:

1. Actual experimental observation.
2. Minimum crack that can be detected by NDE
3. We directly have a reasonable limit for sigma_1 above (e.g. some strength scale).

For intrinsic cohesive models we don't need any nucleation / propagation criterion (disadvantage -> artificial compliance)



For extrinsic cohesive model, LEFM, …. We need propagation criterion (and nucleation for fragmentation problems)