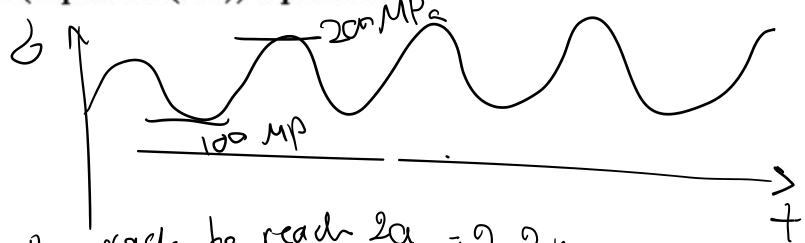
Examples for Fatigue

A large plate contains a crack of length $2a_0$ and is subjected to a constant-amplitude tensile cyclic stress normal to the crack which varies between 100 MPa and 200 MPa. The following data were obtained: for $2a_0 = 2$ mm it was found that $N = 20,000$ cycles were required to grow the crack to $2a_{f_{inal}} = 2.2$ mm, while for $2a = 20$ mm it was found that $N = 1000$ cycles were required to grow the crack to $2a_{f_{inal}} = 22$ mm. The critical stress intensity factor is $K_c = 60$ MPa $\sqrt{m}$. Determine the constants in the Paris (Equation (9.3)) and Formam (Equation (9.4)) equations.

$K = (\sqrt{\pi a})\sigma$

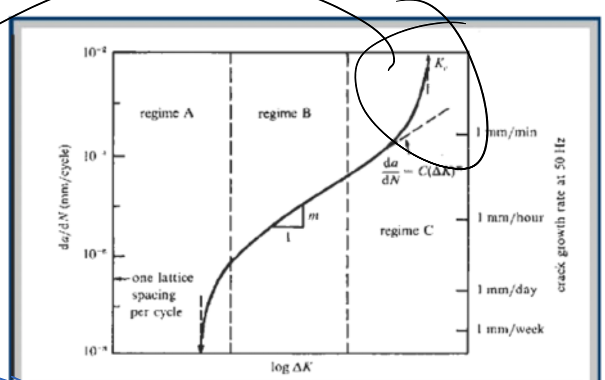


P_1 $\left\{ \begin{array}{l} 2a_0 = 2\,mm \\ \text{it takes } 20{,}000 \text{ cycles for the crack to reach } 2a = 2.2\,mm \end{array} \right.$

P_2 $\left[\begin{array}{l} 2a_0 = 20\,mm \\ \text{" } = 1000 \text{ " " " } 22\,mm \end{array} \right.$

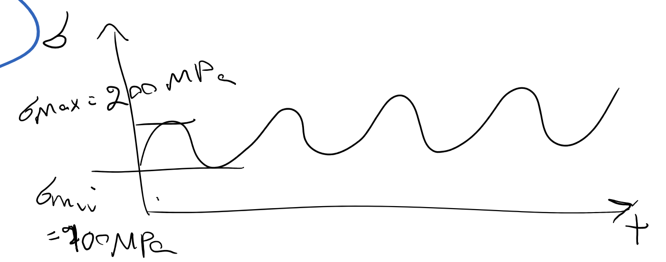
Today with Forman's correction

$$\frac{da}{dN} = \frac{C\,\Delta K^m}{[(1-R)K_c - \Delta K]}$$



(1) $[(1-R)\underbrace{K_c}_{60\,MPa\sqrt{m}} - \Delta K]\frac{da}{dN} = C\,\Delta K^m$

$R = \frac{\sigma_{min}}{\sigma_{max}} = .5$



last time $\Delta K = \Delta\sigma\sqrt{\pi a}$

case a $2a = 2\,mm \longrightarrow \Delta K = 5.6\ MPa\sqrt{m}$ case a ; $\frac{da}{dN} \approx \frac{.1e\text{-}3m}{20000} = 5e\text{-}9$

case b $2a = 20\,mm \longrightarrow \Delta K = 17.72\ MPa\sqrt{m}$ case b da case a $2a = 2\,mm \longrightarrow \Delta K = 5.6\ MPa\sqrt{m}$

case b $2a = 20\,mm \longrightarrow \Delta K = 17.72\ MPa\sqrt{m}$

case a $\frac{da}{dN} \approx \frac{1e\text{-}5\,m}{2000} = 5e\text{-}9$

case b $\frac{da}{dN} = \frac{1\,mm}{1000}$ 1e-6

$$\begin{cases} \text{case a} & [(1-.5)\times 60 - 5.6](5e\text{-}9) = C\,5.6^m \\ \text{case b} & [(1-.5)\times 60 - 17.72](1e\text{-}6) = C\,17.72^m \end{cases}$$

take log of the two sides

$$\begin{cases} -6.914 = \log C + m\,.748 \\ -4.911 = \log C + 1.248\,m \end{cases}$$

log 5.6

last time

$$\boxed{\begin{aligned} C &= 1.22\ \frac{m}{(MPa\sqrt{m})^{4.00}} \\ m &= 4.00 \end{aligned}}$$

With Forman correction

$$\boxed{\begin{aligned} C &= 1.82\ \frac{m}{(MPa\sqrt{m})^{4.6}} \\ m &= 4.6 \end{aligned}}$$

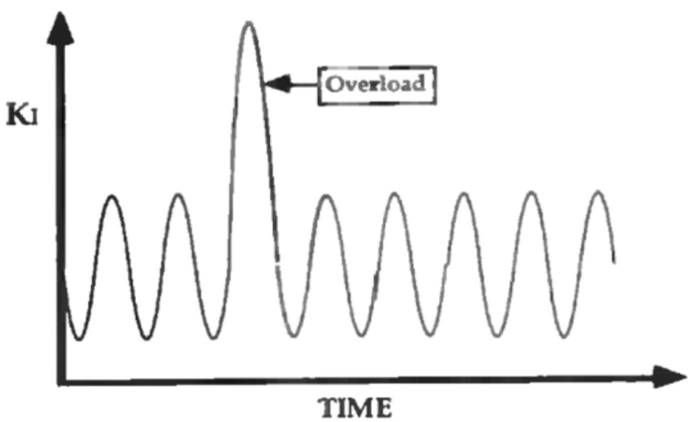
without correction

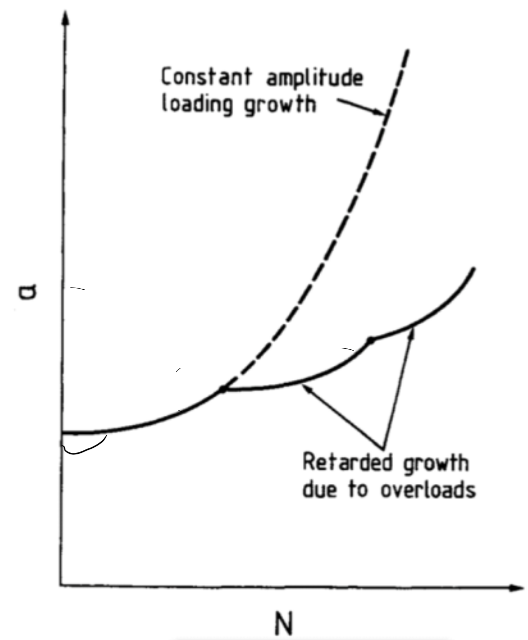
$$\frac{da}{dN} = \frac{C\Delta K^m}{[(1-R)K_c - \Delta K]}$$

$$\frac{da}{dN} = C\,\Delta K^m$$

Overload and crack retardation

It was recognized empirically that the application of a tensile overload in a constant amplitude cyclic loa
leads to crack retardation following the overload; that is, the crack growth rate is smaller than it would have been under constant amplitude loading.





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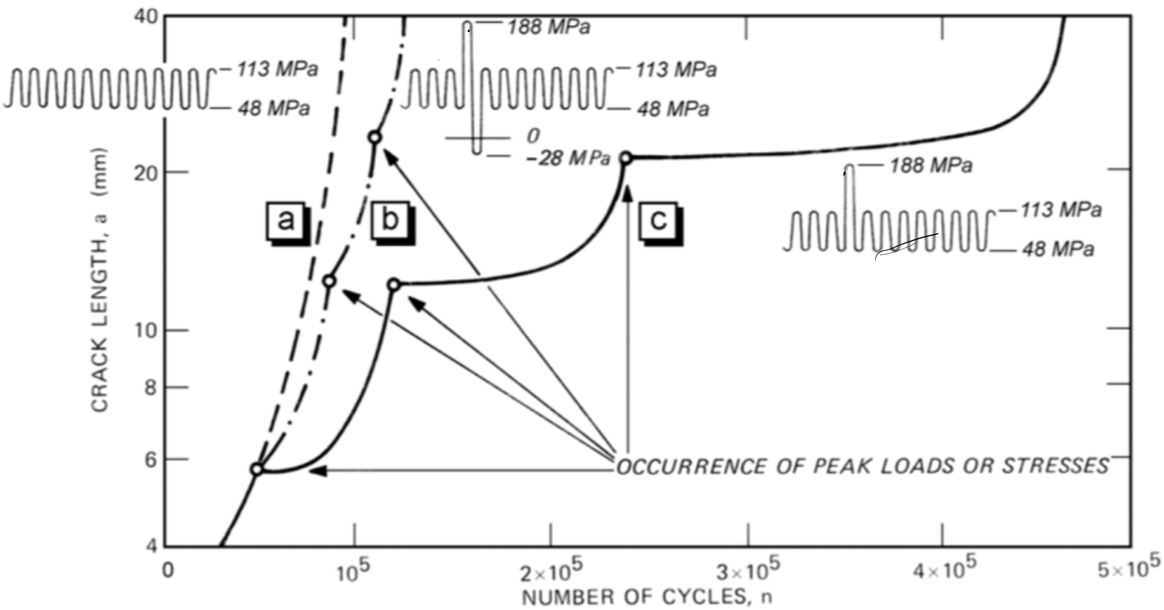
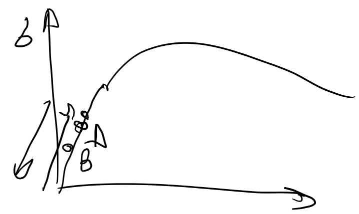
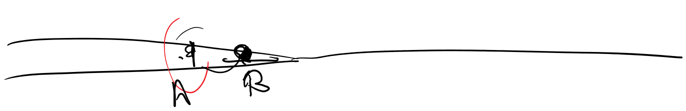


normal fatigue

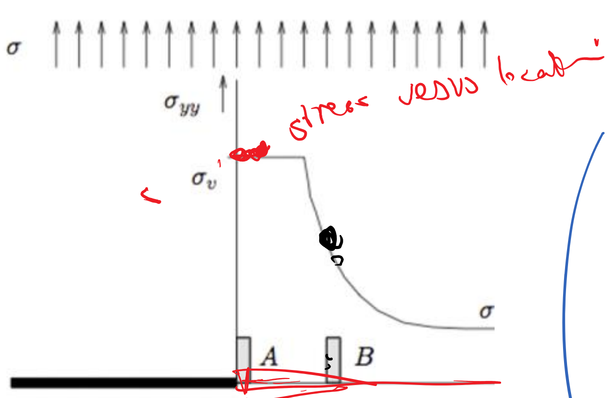


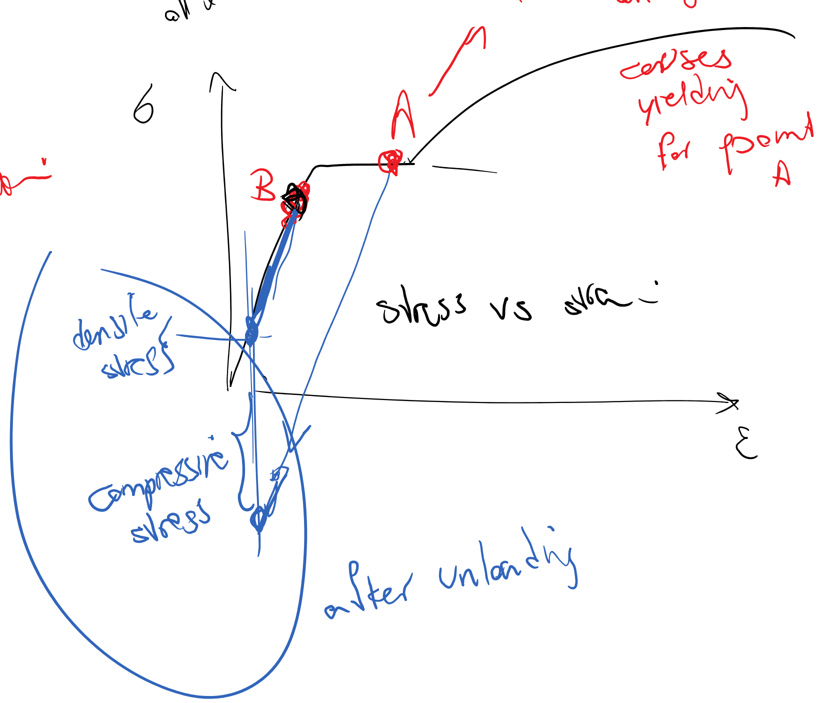


all elastic

load is high enough that causes yielding for point A

What we have an overload?

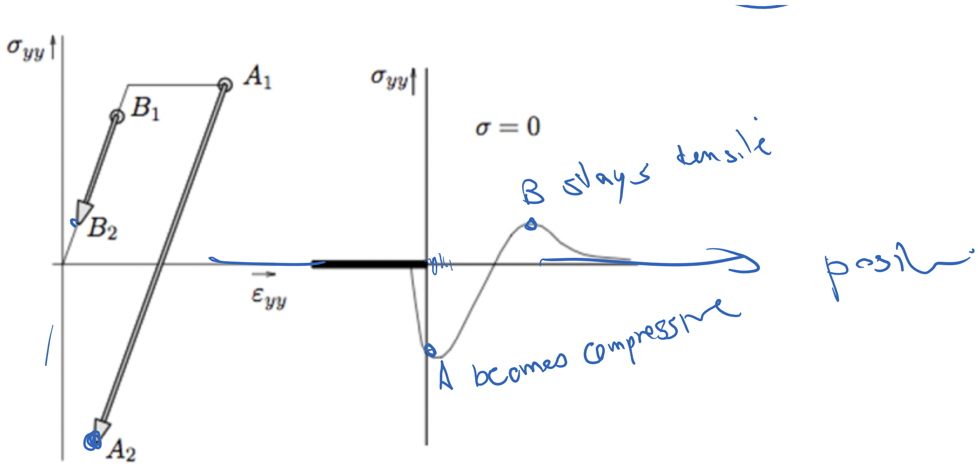




stress vs strain

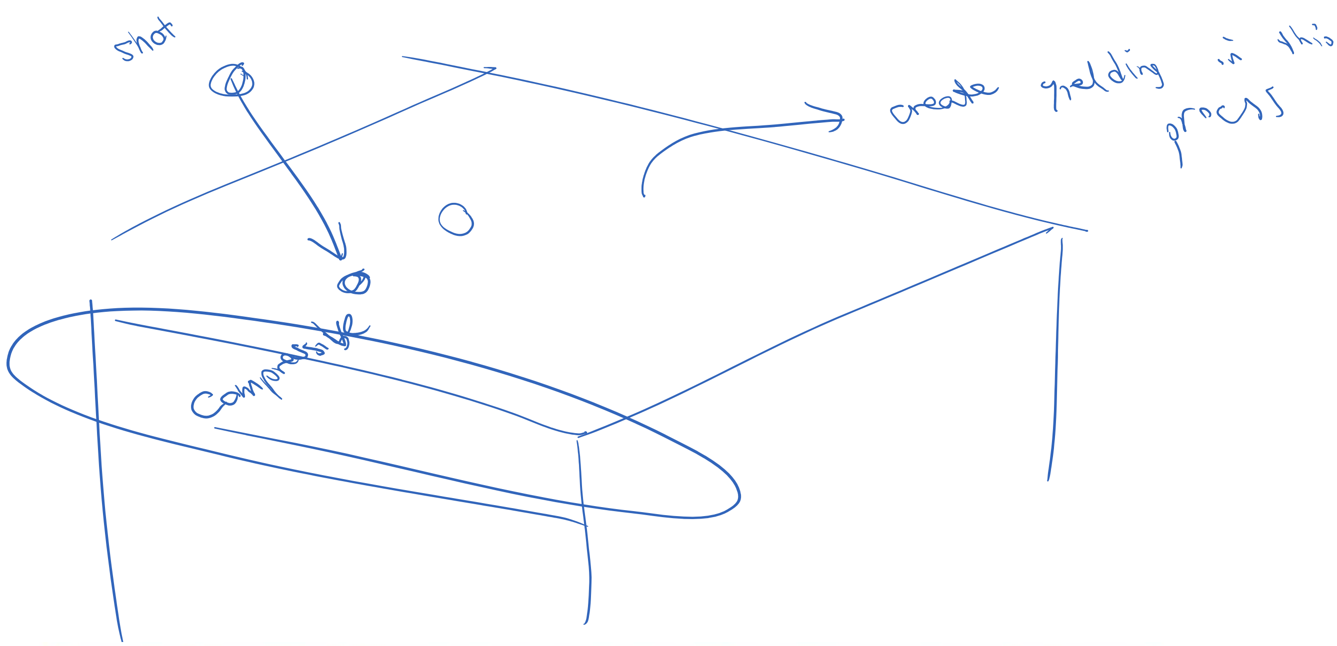
after unloading

Unloading



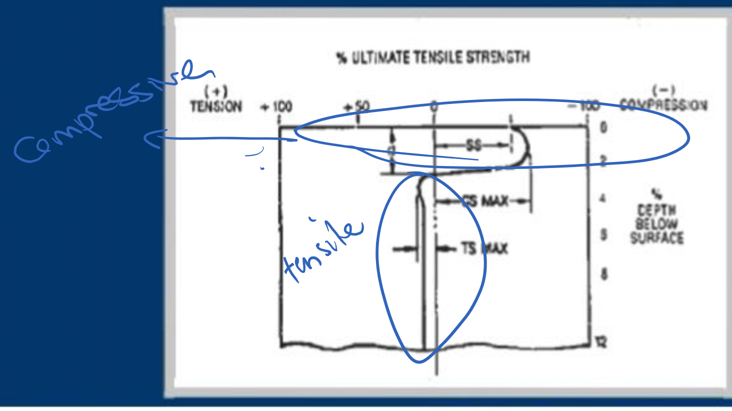
Fatigue crack inhibition: Shot-peening

Shot peening is a cold working process in which the surface of a part is bombarded with small spherical media called *shot*. Each piece of shot striking the surface acts as a tiny peening hammer, imparting to the surface a small indentation or dimple. The net result is a layer of material in a state of residual compression. It is well established that cracks will not initiate or propagate in a compressively stressed zone.



A typical residual stress profile created by shot peening is shown

A typical residual stress profile created by shot peening is shown below:



(source Course presentation Hanlon, S. Suresh MIT)

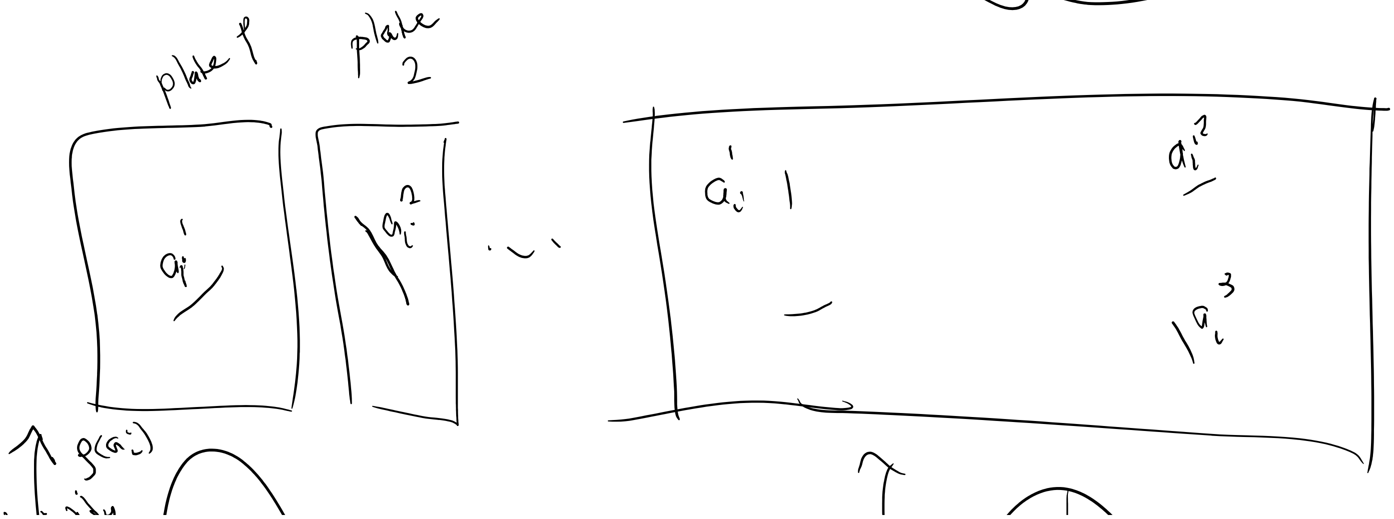
Notes on the final exam:

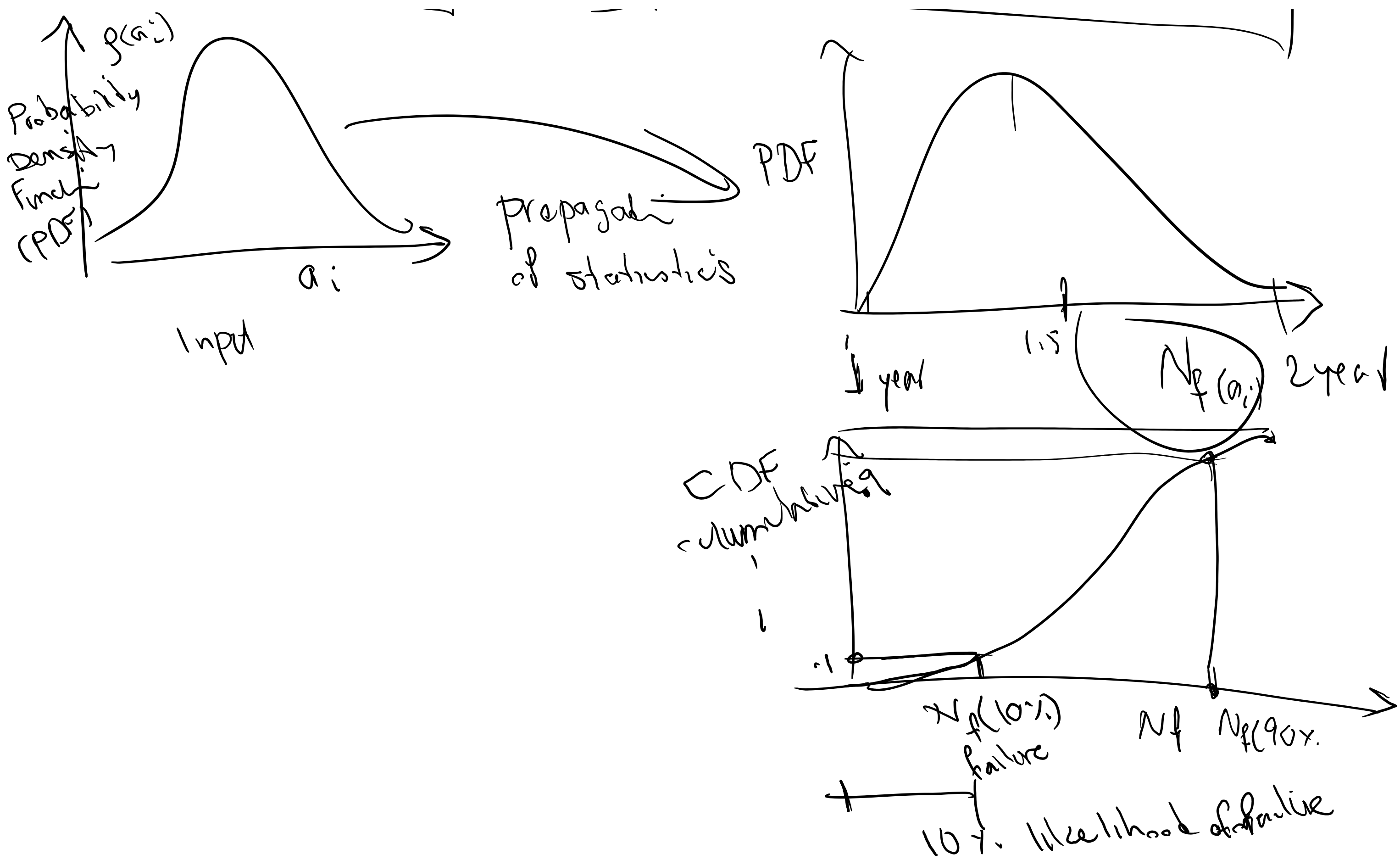
For $m > 2$:

$$N_f = \underbrace{\frac{2}{(m-2)\, C Y^m (\Delta\sigma)^m \pi^{m/2}}}_{D} \left[\frac{1}{(a_i)^{(m-2)/2}} - \frac{1}{(a_f)^{(m-2)/2}} \right]$$

$$\sigma_{max} Y \sqrt{\pi a_f} = K_{IC}$$

$$N_f(a_i) = D\left(\frac{1}{a_i^{\frac{m}{2}-1}} - \frac{1}{a_f^{\frac{m}{2}-1}} \right)$$

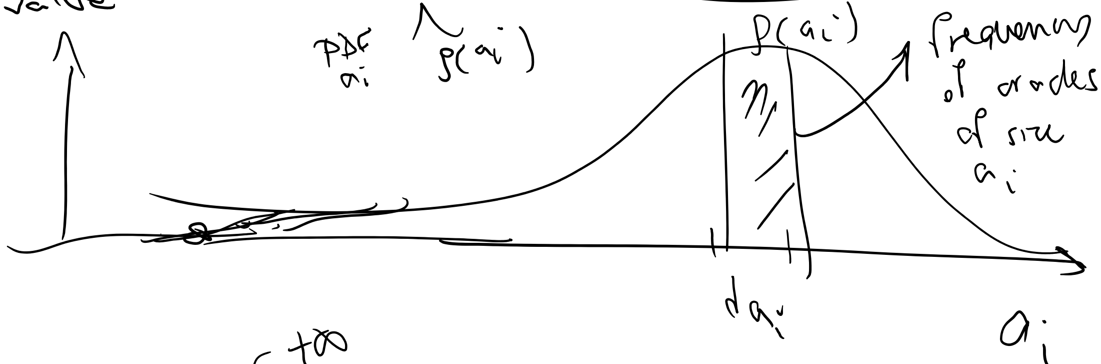




$$N(a_i) = D\left[\frac{1}{a_i^{\frac{m}{2}-1}} - \frac{1}{a_f^{\frac{m}{2}-1}}\right]$$

$$E(f(a_i)) = \int_{-\infty}^{+\infty} f(a_i)\,\rho(a_i)\,da_i$$

↳ expected value



$$E(a_i) = \int^{+\infty} a_i\,\rho(a_i)\,da_i$$

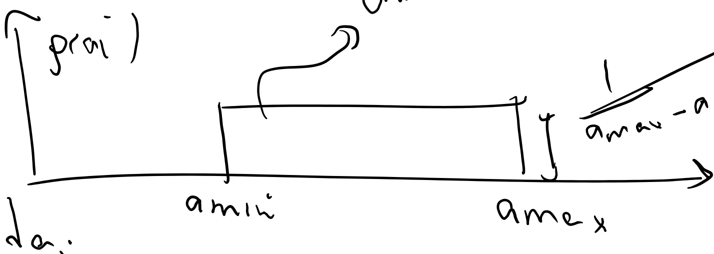
$$E(a_i) = \int_{-\alpha} a_i \, \underline{\rho(a_i)} \, da_i$$

$$\rho = 1$$

PDF_{a_i} $\rho(a_i)$

uniform distribution

$\frac{1}{a_{max} - a_{min}}$



$$E(a_i) = \int_{a_{min}}^{a_{max}} \frac{a_i \, da_i}{a_{max} - a_{min}} = \frac{a_{min} + a_{max}}{2}$$

$$E(N_f(a_i)) = \int \underbrace{D\left(\frac{1}{a_i^{\frac{m}{2}-1}} - \frac{1}{a_f^{\frac{m}{2}-1}}\right)}_{N_f(a_i)} \underbrace{\frac{1}{a_{max} - a_{min}}}_{\rho(a_i)} da_i$$

$$E(N_f(a_i)) \neq \underbrace{N_f(\underbrace{E(a_i)}_{\frac{a_{min} + a_{max}}{2}})}_{\text{Basically what we do with deterministic models}}$$

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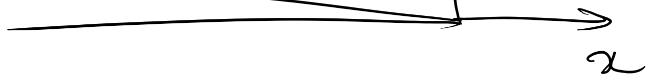
9. Dynamic fracture mechanics and rate effects

Freund's book

Dynamic stress intensity factor



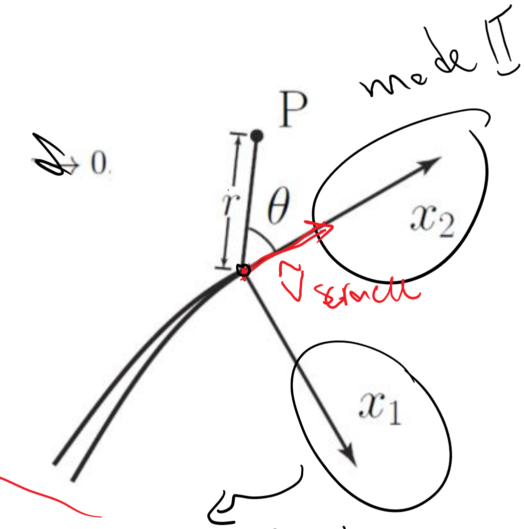
I prefer the following notation

x notation

$x_1 \longrightarrow$ mode I

$x_2 \longrightarrow$ " II

$\hat{v} \to 0$



mode I

Quasi-static Stress Intensity Factors

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$
$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$
$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\frac{3\theta}{2}$$

$\Sigma^{I}_{11}(\theta, \hat{v}=0)$ dynamic effects

quasi-static

$$\sigma_{ij} = \frac{K_I \, \Sigma^{I}_{ij}(\theta, \hat{v})}{\sqrt{2\pi r}} + \frac{K_{II} \, \Sigma^{II}_{ij}(\theta, \hat{v})}{\sqrt{2\pi r}}$$

$\hat{v}$ crack speed

moves along x_2

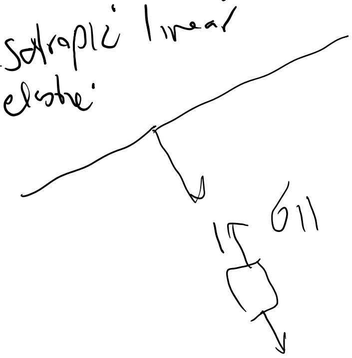
$$s^{ij}(r,\theta,t) = \frac{K_I(t)}{\sqrt{2\pi r}}\Sigma^{ij}_I(\theta, \hat{v}) + \frac{K_{II}(t)}{\sqrt{2\pi r}}\Sigma^{ij}_{II}(\theta, \hat{v}) \quad \text{as} \quad r \to 0.$$

isotropic linear elastic

$$\Sigma^{11}_I = -\frac{1}{D}\left\{(1+\alpha_{II}^2)^2\frac{\cos\frac{1}{2}\theta_I}{\sqrt{\gamma_I}} - 4\alpha_I\alpha_{II}\frac{\cos\frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}}\right\},$$
$$\Sigma^{12}_I = \frac{2\alpha_I(1+\alpha_{II}^2)}{D}\left\{\frac{\sin\frac{1}{2}\theta_I}{\sqrt{\gamma_I}} - \frac{\sin\frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}}\right\},$$
$$\Sigma^{22}_I = \frac{1}{D}\left\{(1+\alpha_{II}^2)(1+2\alpha_I^2-\alpha_{II}^2)\frac{\cos\frac{1}{2}\theta_I}{\sqrt{\gamma_I}} - 4\alpha_I\alpha_{II}\frac{\cos\frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}}\right\}.$$

Mode I

mode I



$$\Sigma^{11}_{II} = \frac{2\alpha_{II}(1+\alpha_{II}^2)}{D}\left\{\frac{\sin\frac{1}{2}\theta_I}{\sqrt{\gamma_I}} - \frac{\sin\frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}}\right\},$$
$$\Sigma^{12}_{II} = \frac{1}{D}\left\{4\alpha_I\alpha_{II}\frac{\cos\frac{1}{2}\theta_I}{\sqrt{\gamma_I}} - (1+\alpha_{II}^2)^2\frac{\cos\frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}}\right\},$$
$$\Sigma^{22}_{II} = -\frac{2\alpha_{II}}{D}\left\{(1+2\alpha_I^2-\alpha_{II}^2)\frac{\sin\frac{1}{2}\theta_I}{\sqrt{\gamma_I}} - (1+\alpha_{II}^2)\frac{\sin\frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}}\right\}$$

Mode II

α's

$$\alpha_{(k)} = \sqrt{1 - \left(\frac{\hat{v}}{C_{(k)}}\right)^2}$$

$\hat{v} \to$ crack speed

$C_{(k)} \to$ elastic wave speeds

$k=1$ mode I

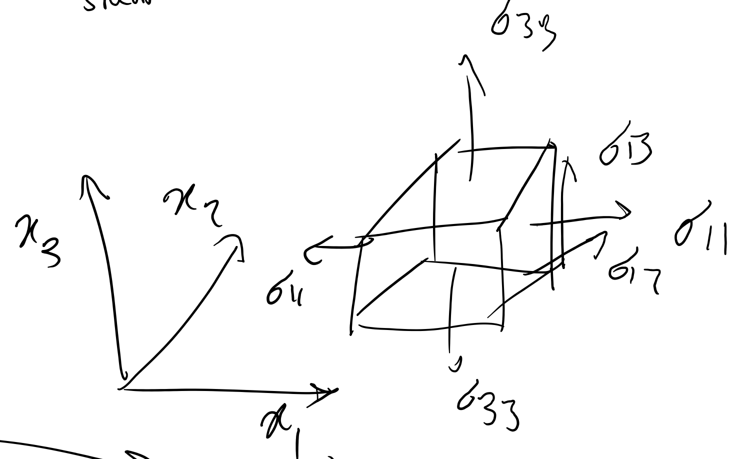
$k=2$ mode II

$$C_{(1)} = C_d = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad c_{(2)} = c_s = \sqrt{\frac{\mu}{\rho}}$$

C_d: dilatational, longitudinal, pressure

c_s: shear

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & 0_{3\times3} & \\ & \lambda+2\mu & \lambda & & & \\ & & \lambda+2\mu & & & \\ & \text{sym} & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \end{Bmatrix}$$



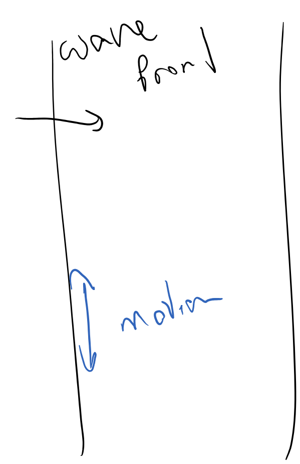
Voigt stiffness matrix

μ → shear modulus (G)

$(E, \nu) \longleftrightarrow (\lambda, \mu)$

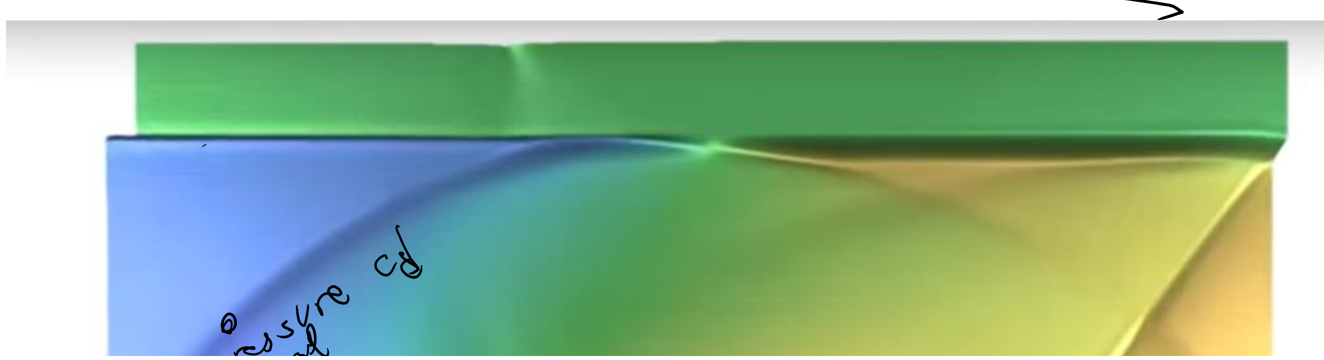
Solid Mechanics C_d

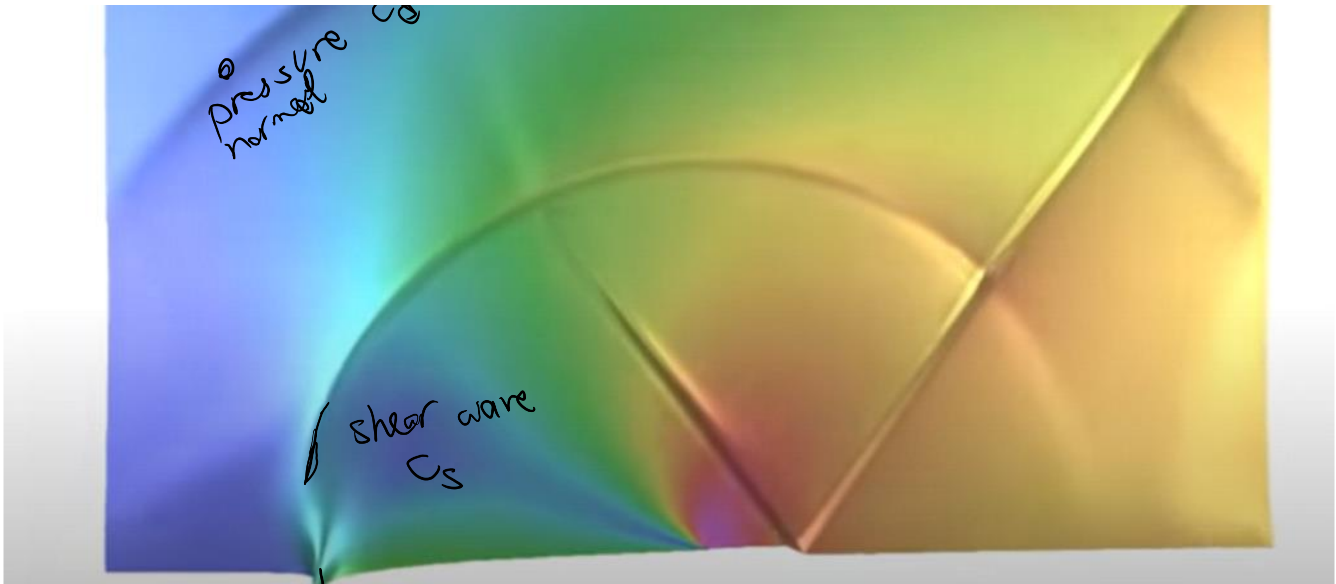
Acoustic



C_s

Electromagnetics





Rayleigh wave $C_R < C_s$

//

$$\Sigma_I^{11} = -\frac{1}{D}\left\{(1+\alpha_{II}^2)^2\frac{\cos\frac{1}{2}\theta_I}{\sqrt{\gamma_I}} - 4\alpha_I\alpha_{II}\frac{\cos\frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}}\right\},$$

$$\Sigma_I^{12} = \frac{2\alpha_I(1+\alpha_{II}^2)}{D}\left\{\frac{\sin\frac{1}{2}\theta_I}{\sqrt{\gamma_I}} - \frac{\sin\frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}}\right\},$$

$$\Sigma_I^{22} = \frac{1}{D}\left\{(1+\alpha_{II}^2)(1+2\alpha_I^2-\alpha_{II}^2)\frac{\cos\frac{1}{2}\theta_I}{\sqrt{\gamma_I}} - 4\alpha_I\alpha_{II}\frac{\cos\frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}}\right.$$

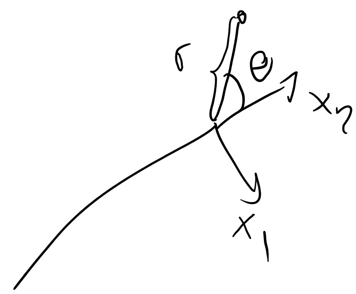
Mode I

$$\Sigma_{II}^{11} = \frac{2\alpha_{II}(1+\alpha_{II}^2)}{D}\left\{\frac{\sin\frac{1}{2}\theta_I}{\sqrt{\gamma_I}} - \frac{\sin\frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}}\right\},$$

$$\Sigma_{II}^{12} = \frac{1}{D}\left\{4\alpha_I\alpha_{II}\frac{\cos\frac{1}{2}\theta_I}{\sqrt{\gamma_I}} - (1+\alpha_{II}^2)^2\frac{\cos\frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}}\right\},$$

$$\Sigma_{II}^{22} = -\frac{2\alpha_{II}}{D}\left\{(1+2\alpha_I^2-\alpha_{II}^2)\frac{\sin\frac{1}{2}\theta_I}{\sqrt{\gamma_I}} - (1+\alpha_{II}^2)\frac{\sin\frac{1}{2}\theta_{II}}{\sqrt{\gamma_{II}}}\right\}$$

Mode II



$$\gamma_{(k)} = \sqrt{1 - \left(\frac{\hat{v}\sin\theta}{c(k)}\right)^2} \qquad \alpha_{(k)} = \sqrt{1 - \left(\frac{\hat{v}}{c(k)}\right)^2}$$

$$\theta_{(k)} = \alpha_{(k)} \operatorname{tg} \theta$$

$$D(\hat{v}) = 4\alpha_I(\hat{v})\,\alpha_{II}(\hat{v}) - (1+\alpha_{II}^2(\hat{v}))^2$$

at Rayleigh wave speed ($\hat{v} = c_R$)

$D = 0$ $\qquad$ ($D(c_R) = 0$)

$D = 0$ $\quad$ ($\boxed{D(G_R) = 0}$)