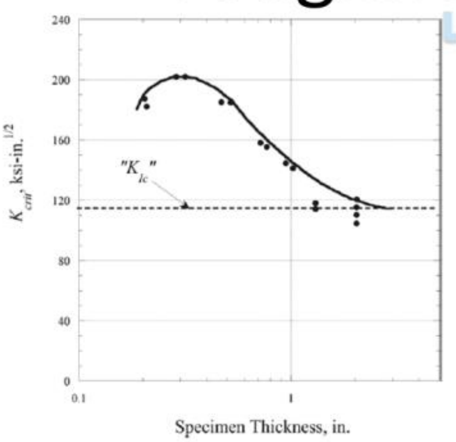
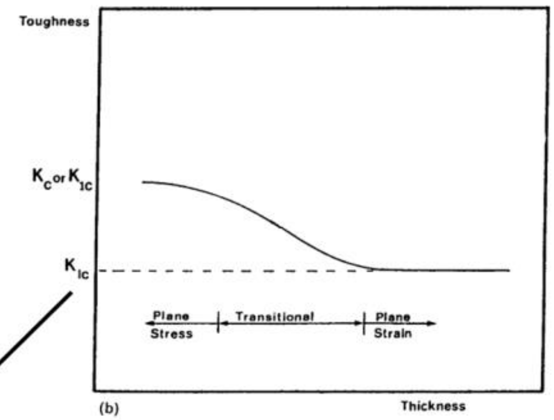
FM20241016

Wednesday, October 16, 2024 9:46 AM

Toughness vs. thickness





Plane strain fracture toughness **lowest K**

(safe)

(Irwin) $K_c = K_{Ic}\left(1 + \frac{1.4}{B^2}\left[\frac{K_{Ic}}{\sigma_{ys}}\right]^4\right)^{1/2}$ Note that $\frac{1}{B^2}\left[\frac{K}{\sigma_{ys}}\right]^4 \propto \left(\frac{r_p}{B}\right)^2$

$(r_p/B)^2$

if $K = K_{Ic}$ corresponding $\underline{r_p} = \left(\frac{K_{Ic}}{\sigma_{ys}}\right)^2$

$\downarrow$

plane strain critical SIF

as $r_p/B \to 0$ ($B \to \infty$; p. strain) $K_c \to K_{Ic}$

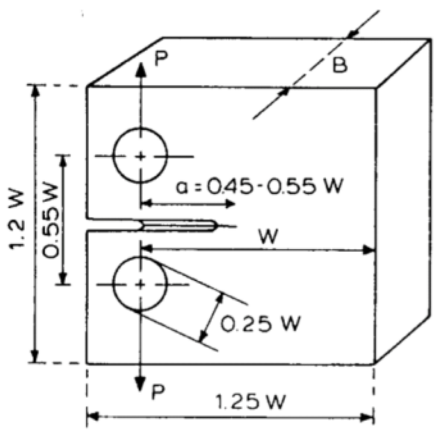
Fracture toughness tests

- Prediction of failure in real-world applications: need the value of fracture toughness
- Tests on cracked samples: PLANE STRAIN condition!!!

Compact Tension Test

$$K_I = \frac{P}{B\sqrt{W}} \frac{\left(2+\frac{a}{W}\right)\left[0.886 + 4.64\frac{a}{W} - 13.32\left(\frac{a}{W}\right)^2 + 14.72(\left(\frac{a}{W}\right)^3 - 5.6\left(\frac{a}{W}\right)\right]}{\left(1-\frac{a}{W}\right)^{3/2}}$$

$a, B, (W-a) \geq 2.5\left(\frac{K_{Ic}}{\sigma_{YS}}\right)^2$



ASTM (based on Irwin's model) for plane strain condition:

$$\boxed{a, B, (W-a) \geq 2.5\left(\frac{K_{Ic}}{\sigma_{Y}}\right)^2}$$

p. strain toughness

5.3. J Integral (Rice 1958)

	LEFM (good when SSY holds)	PFM & NLFM
Global perspective "energy"	$G = -\frac{d\Pi}{dA}$ energy release rate crack growth $G = R$	$G = J$ J integral
Local perspective "stress",...	$\sigma_{ij}^I = \frac{K_I}{\sqrt{2\pi r}} f_{ij}^I(\theta)$ $\sigma_{ij}^I = \frac{\sqrt{GE'}}{\sqrt{2\pi r}} f_{ij}^I(\theta)$ pure mode I	$\sigma_{yy} = J^\alpha r^\beta$
relationship	$G = K_I^2 + K_{II}^2$ ...	

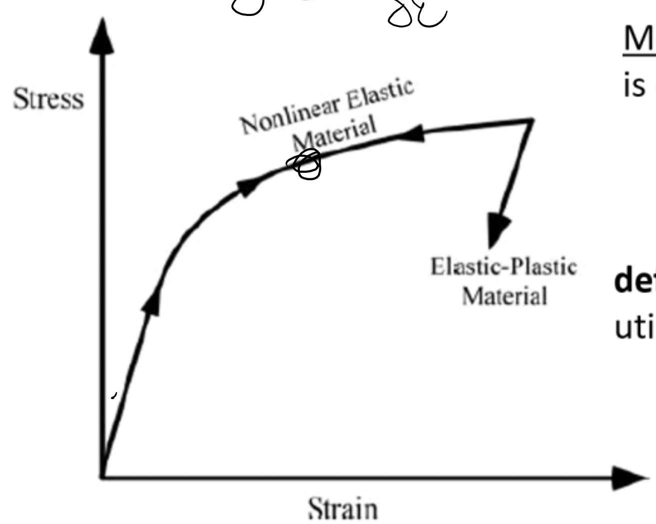
relationship

$$G = \frac{K_I^2 + K_{II}^2}{E'}$$ in plane

Idea

Replace complicated plastic model with nonlinear elasticity (no unloading)

$$\sigma = \frac{\partial W}{\partial \varepsilon}$$



<u>Monotonic loading</u>: an elastic-plastic mate is equivalent to a nonlinear elastic material

deformation theory of plasticity can be utilized

For now Linear Elasticity $\longrightarrow$ Nonlinear Elasticity

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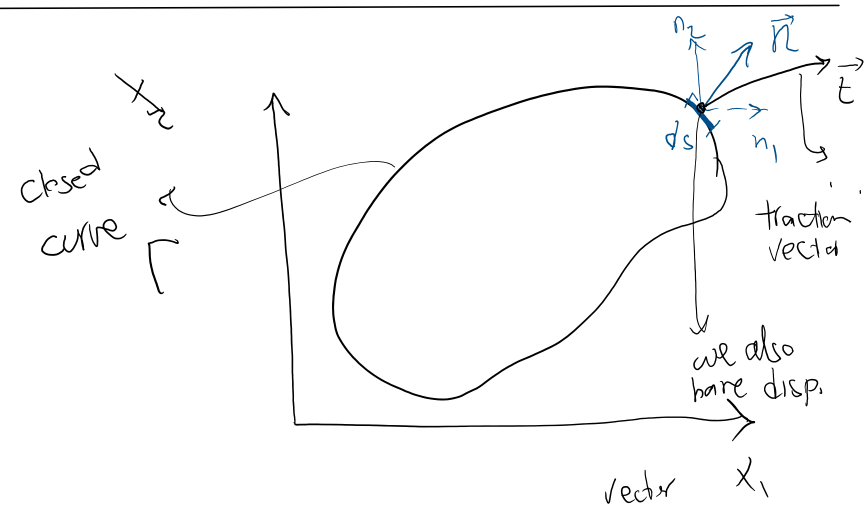
NL elasticity

Eshelby, Cherepanov 1967

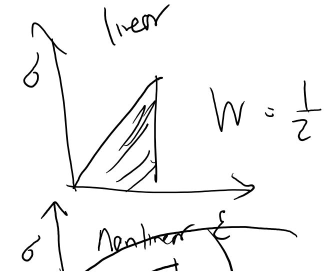
$$J_k = \int_\Gamma \left(W n_k - \vec{t} \cdot \frac{\partial \vec{u}}{\partial x_k} \right) ds$$

for k = 1 & 2

W: strain energy density

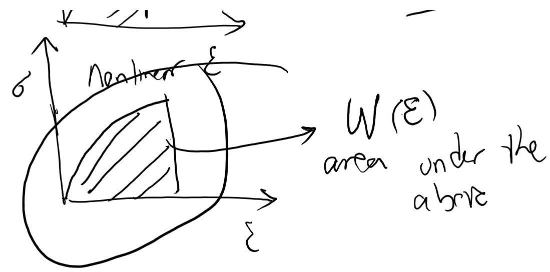


1D



$$W = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}E\varepsilon^2$$

W : strain energy density



$$\boxed{\sigma(\varepsilon) = \frac{dW(\varepsilon)}{d\varepsilon}}$$

$k=1$

$$J_1 = \int_\Gamma \left(W n_1 - \vec{t} \cdot \frac{\partial \vec{u}}{\partial x_1} \right) ds$$

$$J_2 = \int \left(W n_2 - \vec{t} \cdot \frac{\partial \vec{u}}{\partial x_2} \right) ds$$

2D/3D linear $\quad W = \frac{1}{2} \sigma : \varepsilon = \frac{1}{2} \varepsilon : C \varepsilon$

in general $\quad = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_{ij} \varepsilon_{ij}$

$$\boxed{\sigma = \frac{\partial W(\varepsilon)}{\partial \varepsilon} \qquad \sigma_{ij} = \frac{\partial W(\varepsilon)}{\partial \varepsilon_{ij}}}$$

Eshelby and Cherpanov showed that this integral (J1 and J2) is zero over closed path Γ for smooth solutions of **elastostatic** problems with **no body force**.

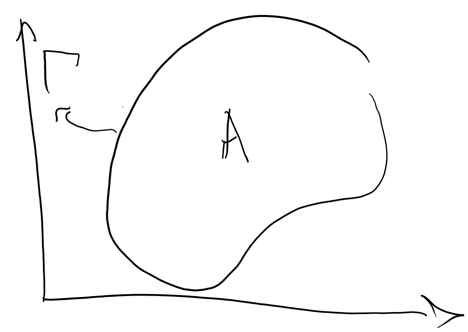
Equation of motion $\quad \nabla \cdot \sigma + \rho \cancel{b} = \rho \cancel{\ddot{u}}$

0 $\qquad$ 0 $\qquad$ acceleration

$$\boxed{\nabla \cdot \sigma = 0 \quad \text{elastostatic, no body force}} \quad ★$$

1) Proof of why J = 0 over a closed path:

$$J_k = \int_\Gamma \left(W(\varepsilon) n_k - \underbrace{t_i \cdot \frac{\partial u_i}{\partial x_k}}_{\vec{t} \cdot \frac{\partial \vec{u}}{\partial x_k}} \right) ds$$



FYI $a_i b_i$

dimension

Γ

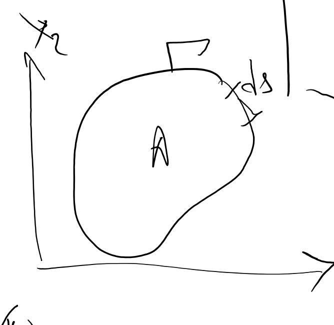
$$\vec{t} \cdot \frac{\partial \vec{u}}{\partial x_k}$$

$$t_i = \sigma_{ij} n_j \qquad \vec{t} = \sigma n$$

$$J_k = \int_\Gamma \left[W(\varepsilon) n_k - \underbrace{(\sigma_{ij} n_j)}_{t_i} \frac{\partial u_i}{\partial x_k} \right] ds$$

We'll use divergence theorem

$$\boxed{J_k = \int_A \left[\underbrace{\frac{\partial W}{\partial x_k}}_{(1)} - \underbrace{\frac{\partial}{\partial x_j}\left(\sigma_{ij} \frac{\partial u_i}{\partial x_k} \right)}_{(2)} \right] dV} \quad (1)$$



FYI $a_i b_i$

dimension

$$= \sum_{i=1}^{d} a_i b_i$$

$$t_i \frac{\partial u_i}{\partial x_k} = \sum_{i=1}^{2} t_i \frac{\partial u_i}{\partial x_k}$$

$$t_i = \sigma_{ij} n_j = \sum_{j=1}^{2} \sigma_{ij} n_j$$

$$\int_\Gamma f n_i \, ds = \int_A \frac{\partial f}{\partial x_i} dV \qquad f_{,i}$$

general form of Div theorem

(1)

$$W_{,k} = \frac{\partial W(\varepsilon)}{\partial x_k} = \underbrace{\frac{\partial W(\varepsilon)}{\partial \varepsilon_{mn}}}_{\sigma_{mn}} \frac{\partial \varepsilon_{mn}}{\partial x_k} =$$

$$\sigma_{mn} \left(\frac{\partial \left(\frac{u_{m,n} + u_{n,m}}{2} \right)}{\partial x_k} \right) = \sigma_{mn} \left(\frac{u_{m,nk} + u_{n,mk}}{2} \right)$$

we know $\sigma_{mn} = \sigma_{nm}$ (Stress tensor is symmetric)

$$\Longrightarrow \boxed{W_{,k} = \sigma_{mn} u_{m,nk}} \quad \text{term I}$$

(2) $$\frac{\partial}{\partial x_j}\left(\sigma_{ij} \frac{\partial u_i}{\partial x_k} \right) = \frac{\partial \sigma_{ij}}{\partial x_j} \frac{\partial u_i}{\partial x_k} + \sigma_{ij} \frac{\partial^2 u_i}{\partial x_j \partial x_k}$$

$(\nabla\cdot\sigma)_i = 0$

equation of motion ($\ddot{u}$ & $b=0$)

$$\boxed{\frac{\partial}{\partial x_j}\left(\sigma_{ij}\frac{\partial u_i}{\partial x_k}\right) = \sigma_{ij}\frac{\partial^2 u_i}{\partial x_j \partial x_k}} \quad \text{term 2}$$

plug terms 1 & 2 in equation (1)

$$J_k = \int_A \Big[\underbrace{\frac{\partial W}{\partial x_k}}_{(1)} - \underbrace{\frac{\partial}{\partial x_j}\left(\sigma_{ij}\frac{\partial u_i}{\partial x_k}\right)}_{(2)} \Big] dV \qquad (1)$$

$$J_k = \int_A \left(\sigma_{mn}\, u_{m,nk} - \sigma_{ij}\, u_{i,jk}\right) dV \qquad i \to m,\ j \to n$$

$$= \int_A \left(\sigma_{mn}\, u_{m,nk} - \sigma_{mn}\, u_{m,nk}\right) dV = 0$$

$$\boxed{J_k = 0}$$

Recap: Eshelby and Cherpanov showed that Jk = 0 ovr closed curves for C1 smooth solutions

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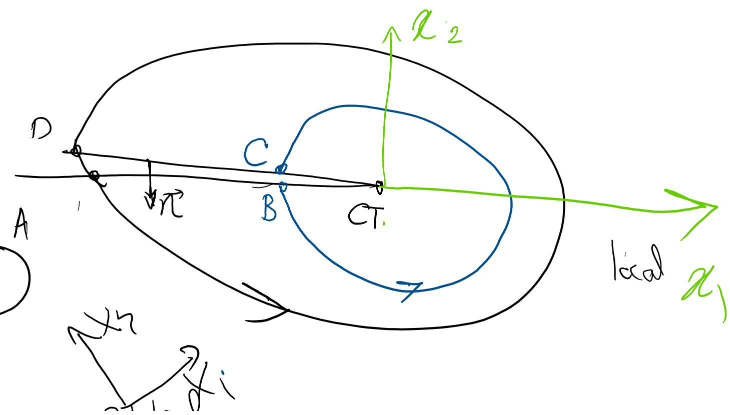
Second part (2):

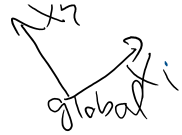
We'll show that J integral is equal over any closed curve around the crack tip

$$J_{AD} = \int_A^D (\text{integrand})\, ds$$



$$J_{AD} = J_{BC} \neq 0$$

we're doing this for J_1



$$J_{ADCBA} = 0 = J_{AD} + \underbrace{\cancel{J_{DC}}}_{0} + J_{CB} + \underbrace{\cancel{J_{BA}}}_{0}$$

why these two are zero

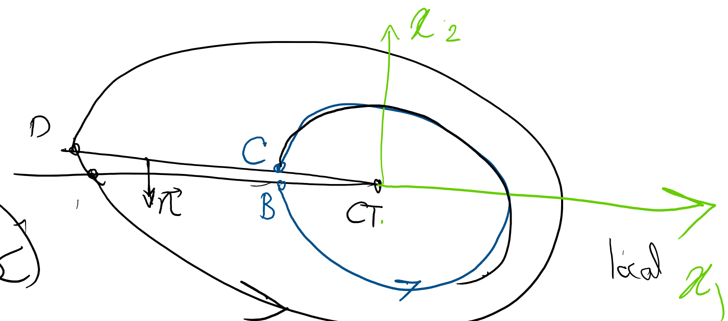
$$J_{DC} = \int_C^D \left(W \cancel{n_1}^{\,0} - \vec{t} \cdot \frac{\partial \vec{u}}{\partial x_1} \right) ds = 0$$

$= 0$ if we have TRACTION FREE CRACK SURFACE

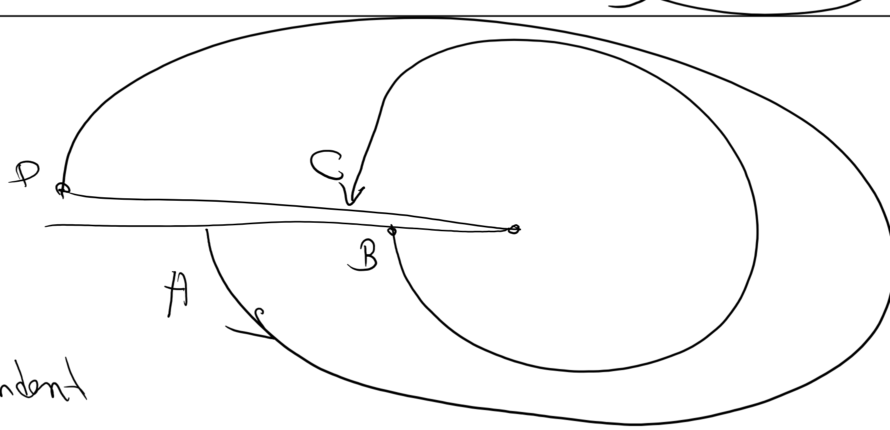
Similarly $J_{BA} = 0$

$$J_{AD} + J_{CB} = 0$$

$$\Rightarrow J_{AD} = -J_{CB} = -(-J_{BC})$$



$J_{AD} = J_{BC}$



path independent value of J around a crack

Last step, shown by Rice,

$$J = G \qquad (J_{AD} = J_{BC} = G)$$

Why J = G?
https://rezaabedi.com/wp-content/uploads/Courses/FractureMechanics/J=G.pdf

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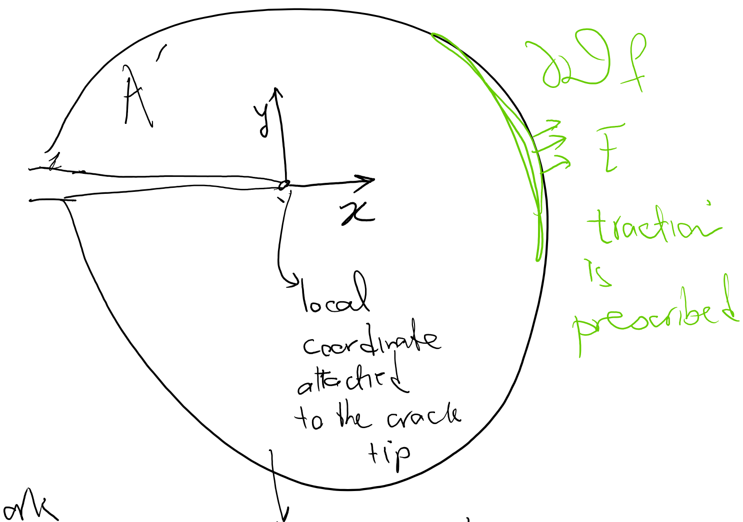
$$G = -\frac{d\Pi}{\underbrace{dA}_{\text{crack surface}}} = -\frac{d\Pi}{B\,d\underbrace{a}_{\text{crack length}}}$$



domain A' around the crack tip

$$\Pi = \underbrace{U_e}_{\text{internal energy}} - \underbrace{W_{ext}}_{\text{external work}}$$

$$U_e = \int_{A'} \underbrace{W(\varepsilon)}_{\text{strain energy density}} dV$$

$$W_{ext} = \int_{A'} u\,\rho \cancelto{0}{b}\, dV + \int_{\partial\Omega_f} u\,\bar{t}\, ds$$

$$\Rightarrow \quad G = -\frac{1}{B}\frac{d}{da}\big(\overbrace{U_e - W_{ext}}^{\Pi}\big) = -\frac{1}{B}\frac{d}{da}\left(\int_{A'} W(\varepsilon)\,dV - \int_{\partial\Omega_f} u\,\bar{t}\,ds\right)$$