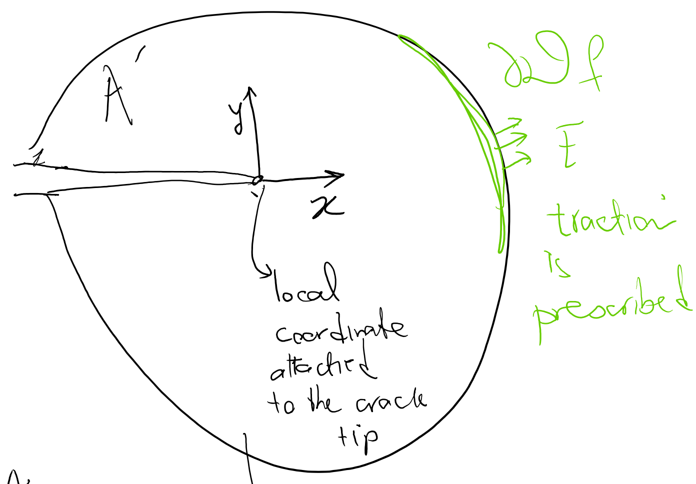
FM20241021

Monday, October 21, 2024 9:40 AM

Why J = G?
https://rezaabedi.com/wp-content/uploads/Courses/FractureMechanics/J=G.pdf

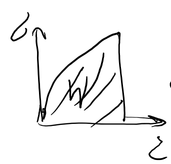
$$G = -\frac{d\Pi}{\underbrace{dA}_{\text{crack surface}}} = -\frac{d\Pi}{B\,\underbrace{da}_{\text{crack length}}}$$



$$\Pi = \underbrace{U_e}_{\text{internal energy}} - \underbrace{W_{ext}}_{\text{external work}}$$

$$U_e = \int_{A'} \underbrace{W(\varepsilon)}_{\text{strain energy density}}\, dV$$

1D



$$\sigma = \frac{\partial W}{\partial \varepsilon}$$

$$W_{ext} = \int_{A'} u\, \cancel{\rho b}\, dV + \int_{\partial\Omega_f} u\bar{t}\, ds$$

$$\Longrightarrow \quad G = -\frac{1}{B}\frac{d}{da}\big(\overbrace{U_e - W_{ext}}^{\Pi}\big) = -\frac{1}{B}\left(\frac{d}{da}\right)\left(\int_{A'} W(\varepsilon)\,dV - \int_{\partial\Omega_f} u\bar{t}\,ds\right)$$

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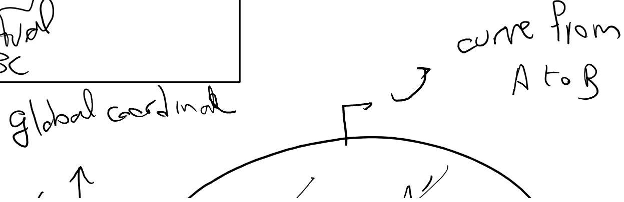
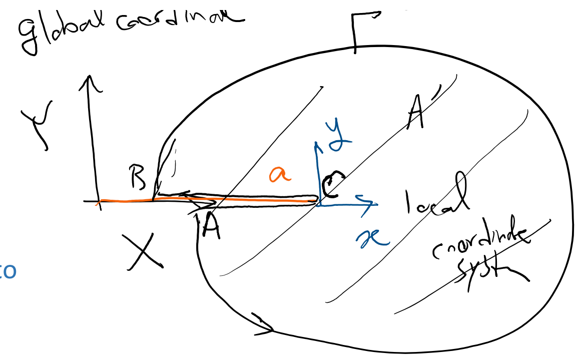
let $B = 1$ for simplicity

$$\boxed{G = -\int_{A'} \frac{d}{da} W(\varepsilon)\,dV + \int_{\underbrace{\partial\Omega_f}_{\text{natural BC}}} \frac{d}{da}(u\bar{t})\,ds} \quad (1)$$



$$\frac{df}{}\Big|$$

$$\frac{df}{da}\Big|_{X \text{ is fixed}}$$



Since the J-integral is computed in the coordinate attached to the crack tip, we need to change the coordinate system used to express terms above and evaluate derivatives in the local coordinate system.

$$\frac{d f(x,a)}{da}\Big|_{X \text{ fixed}} = \frac{\partial f(x,a)}{\partial x}\left(\frac{\partial x}{\partial a}\right) + \frac{\partial f(x,a)}{\partial a}\left(\underbrace{\frac{\partial a}{\partial a}}_{1}\right)$$

$$X = a + x + \underbrace{\text{const}} \Rightarrow$$

$$x = X - a - \text{const} \Rightarrow$$

$$\frac{\partial x}{\partial a} = -1$$

$$\Rightarrow \quad (2) \quad \boxed{\frac{df(x,a)}{da}\Big|_{X \text{ fixed}} = \frac{\partial f(x,a)}{\partial a} - \frac{\partial f(x,a)}{\partial x}}$$

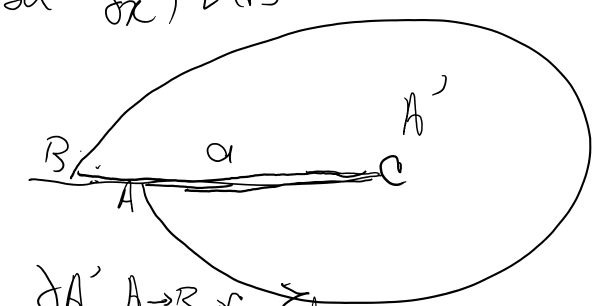
plug (2) in (1) $\longrightarrow$ expanded derivation

$$G = \int_{A'} \frac{d}{da} W(\varepsilon)\, dV - \int_{\partial D_f} \frac{d}{da}(u \cdot t)\, ds \quad (1)$$

equal $\longrightarrow \int_{\partial A'} \left(\frac{du}{da}\right) t \, ds$

$\partial A'$ boundary of A'

$$G = \int_{A'} \left(\frac{\partial W}{\partial a} - \frac{\partial W}{\partial x}\right) + \int_{\partial A'} \left(\frac{\partial u}{\partial a} - \frac{\partial u}{\partial x}\right) t\, ds$$



$$G = I_1 + I_2$$

$\partial A'$ A→B→C→A

$$0 = I_1 = -\int_{A'} \frac{\partial W}{\partial a} dV + \int_{\partial A'} \frac{\partial u}{\partial a}\, t\, ds = 0$$

(see notes)

proof of this is similar to J

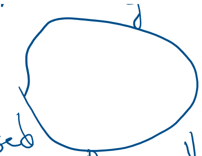
J = 0

[illegible]

$$I_2 = -\int_{A'} \frac{\partial W}{\partial x}\, dV + \int_{\partial A'} \frac{\partial u}{\partial x}\, t\, ds$$

proof of this is ...

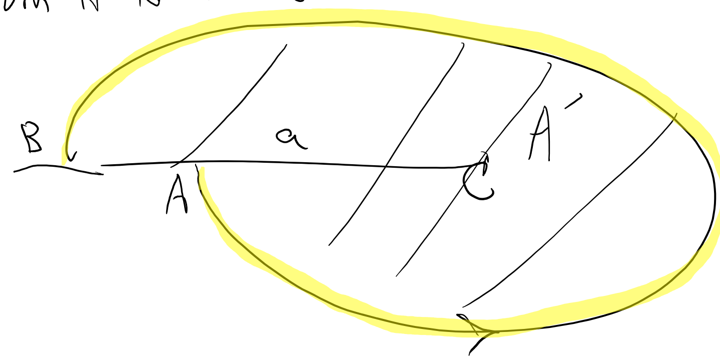
$J = 0$ on a closed curve with smooth smooth



$$\boxed{G = \underbrace{\int_{A'} -\frac{\partial W}{\partial x}\, dV}_{\text{turn it to boundary}} + \int_{\partial A'} \frac{\partial u}{\partial x}\, t\, ds}$$

(3)

want to integral (divergence theorem)

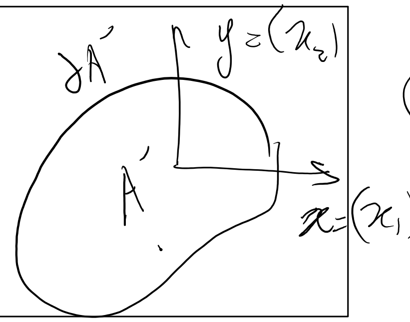


$$\int_{A'} \frac{\partial W}{\partial x}\, dV = \int_{\partial A'} W n_x\, ds \qquad (x \to x_1,\; n_x)$$

(4)

$$\int_{A'} \frac{\partial f}{\partial x_i}\, dV = \int_{\partial A'} f n_i\, ds \Rightarrow$$

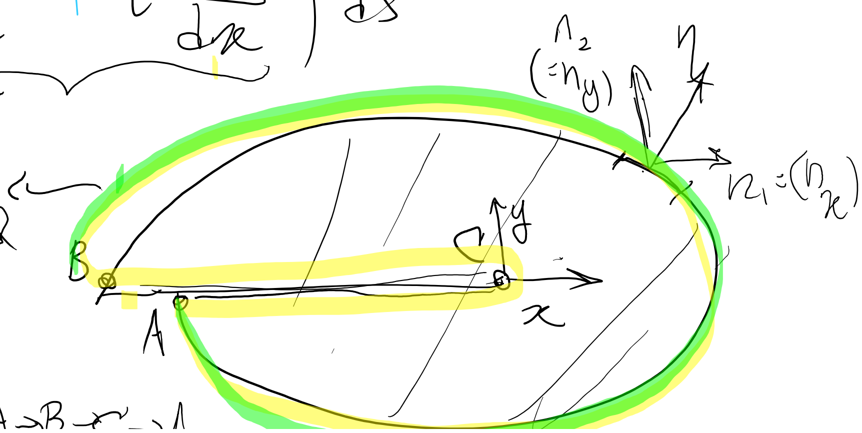
$$\int_{A'} \nabla f\, dV = \int_{\partial A'} f n\, ds$$



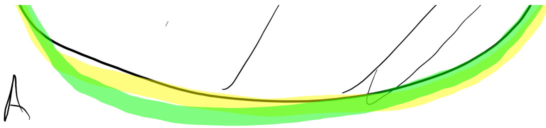
(4)

$$G = \int_{\partial A'} \underbrace{\left(-W n_x + t \frac{du}{dx}\right)}_{J} ds$$

green J integral



$\partial A' = A \to B \to C \to A$



$\partial A' = A \to B \to C \to A$
yellow

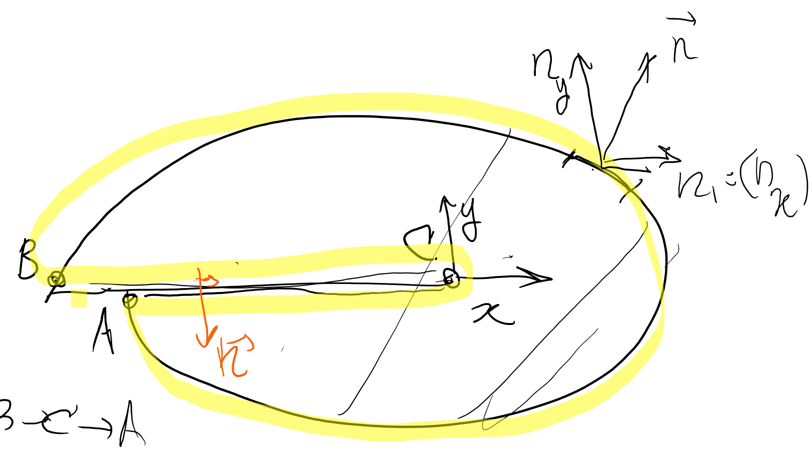
$$G = \int_{\partial A' = A\to B\to C\to A} I\, ds = \underbrace{\int_{A\to B} t\, ds}_{J} + \int_{B\to C} t\, ds + \int_{C\to A} I\, ds$$

$$\boxed{G = J + \int_{B\to C} I\, ds + \int_{C\to A} I\, ds} \qquad (5)$$

$$\int_{B\to C} \left(W n_x + \vec{t}\,\frac{d\vec{u}}{dx}\right) ds$$

$n_x = 0$ on crack surface

crack surface is traction-free ⇒ $\vec{t} = 0$



$\partial A' = A \to B \to C \to A$
yellow

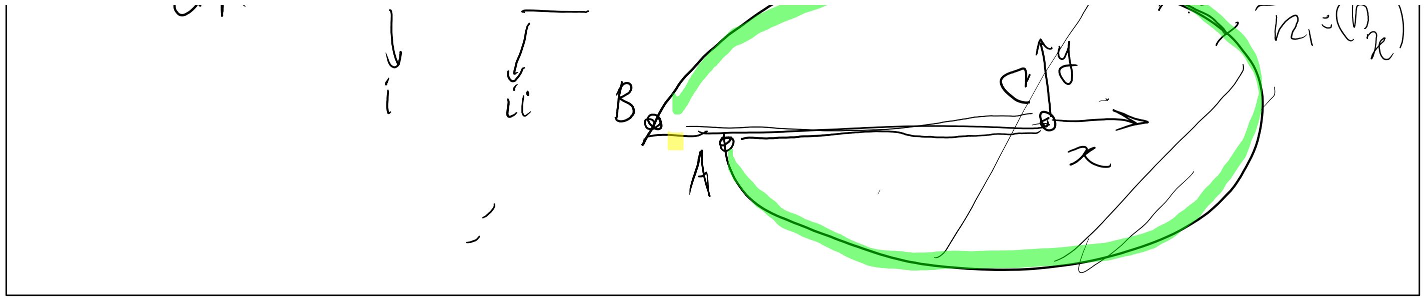
Similarly $\int_{C\to A} I\, ds = 0$

(5) ⇒

$$\boxed{G = J_1 = \int_A^B \Big(\underbrace{W n_x}_{i} + \underbrace{\frac{\partial \vec{u}}{\partial x}\vec{t}}_{ii}\Big) ds}$$

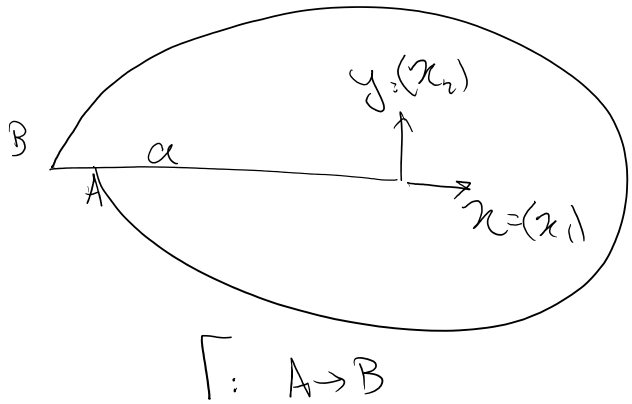
$$G = -\frac{d\Pi}{dA} = -\frac{dU_e}{dA} + \frac{dW_{ext}}{dA}$$

(6)



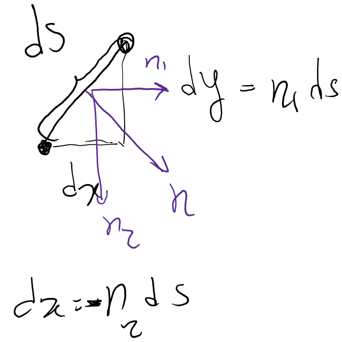
Relation of KI, KII to J1 and J2 (in-plane fracture)

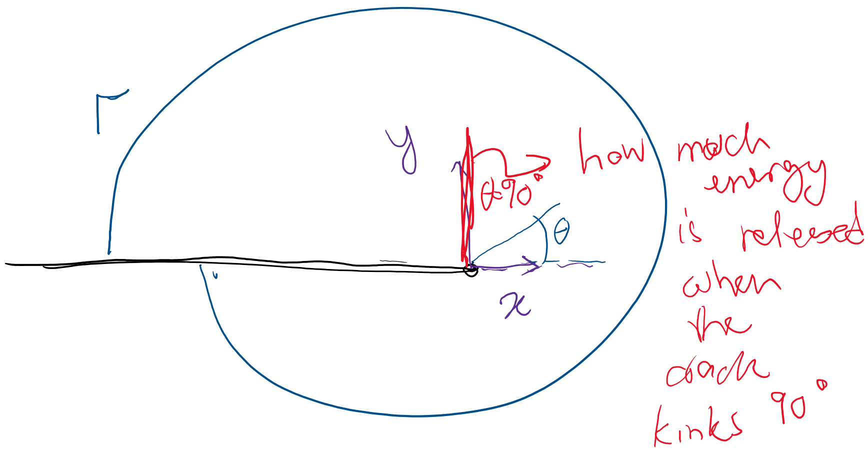
$$J_k = \int_\Gamma \left(-W(\varepsilon)\, n_k + \vec{t}\,\frac{d\vec{u}}{dx_k}\right) ds$$



$$J_1 = \int_\Gamma \left(-W n_1 + \vec{t}\,\frac{du}{dx_1}\right) ds = \int_\Gamma -W\,dy + \vec{t}\,\frac{du}{dx}\,ds$$

$$J_2 = \int_\Gamma \left(-W n_2 + \vec{t}\,\frac{du}{dx_2}\right) ds = \int_\Gamma W\,dx + \vec{t}\,\frac{du}{dy}\,ds$$





$J_1 = G(\theta = 0)$

$J_2 = G(\theta = 90^\circ)$

$$J_1 = \int_\Gamma \left(-W\,dy + \vec{t}\,\frac{du}{dx}\,ds\right) = G(\theta = 0) \quad (7)$$

$$J_2 = \int \left(W\,dx + t\,\frac{du}{dx}\,ds\right) = G\left(\theta = \frac{\pi}{2}\right)$$

$K_I, K_{II} \Longleftrightarrow J_1 \,\&\, J_2$

Recall

$$\boxed{\begin{aligned} J_1 &= G(\theta = 0) = \frac{K_I^2 + K_{II}^2}{E'} \quad (8i) \\ J_2 &= G\left(\theta = \frac{\pi}{2}\right) = -\frac{2K_I K_{II}}{E'} \quad (8ii) \end{aligned}} \quad (8)$$

in-plane fracture

$$E' = \begin{cases} E & \text{p. stress} \\ \dfrac{E}{1-\nu^2} & \text{p. strain} \end{cases}$$

8ii $\quad K_{II} = \dfrac{J_2 E'}{2K_I}$ plug in (8i)

$$K_I^2 + \Bigg(\underbrace{\frac{J_2 E'}{2K_I}}_{K_{II}}\Bigg)^2 = J_1 E'$$

multiply by K_I^2

k^2 $\qquad\qquad\qquad\qquad$ K_I^2

$$\left(K_I^2\right)^2 - J_1 E'\left(K_I^2\right) + \frac{\left(J_2 E'\right)^2}{4} = 0$$

$$z = K_I^2$$

$$z^2 - J_1 E' z + \frac{\left(J_2 E'\right)^2}{4} = 0$$

$$\boxed{K_{I,II} = \pm\sqrt{\frac{E'}{2}\left(J_1 \pm \sqrt{J_1^2 - J_2^2}\right)}} \quad (9)$$

$K_I = a \qquad K_{II} = b$

$K_I = -a \qquad K_{II} = -b$

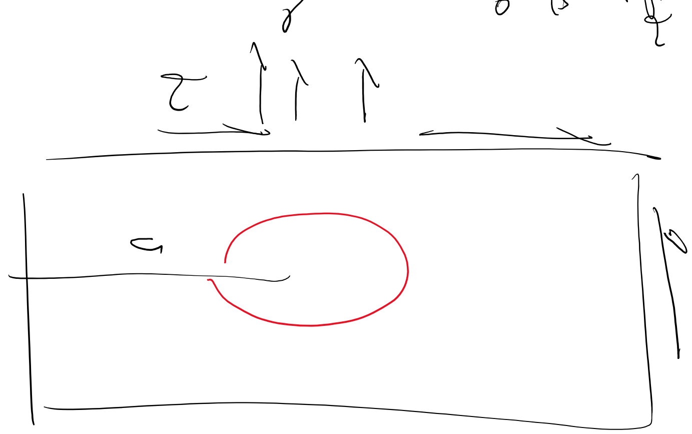
$K_I = b \qquad K_{II} = a$

$K_I = -b \qquad K_{II} = -a$

$$a = \sqrt{\frac{E'}{2}\left(J_1 + \sqrt{J_1^2 - J_2^2}\right)}$$

$$b = \sqrt{\frac{E'}{2}\left(J_1 - \sqrt{J_1^2 - J_2^2}\right)}$$

σ is higher than τ



Choose the correction solution at hand

$J_1 = 109 \qquad E' = 1$

$J_2 = -60$

$$a, b = \pm\sqrt{\frac{J_1 + \sqrt{J_1^2 - J_2^2}}{2E'}}$$

$$= \pm\sqrt{\frac{109 \pm \sqrt{109^2 - (60)^2}}{}} = \begin{matrix}10\\3\end{matrix}$$

$$\boxed{\begin{matrix}K_I = 10\\ K_{II} = 3\end{matrix}}$$

$$= \pm \sqrt{\frac{109 \pm \sqrt{109^2 - (60)^2}}{2}} = \begin{matrix} 10 \\ 3 \end{matrix}$$

$$\boxed{\begin{matrix} K_I = 10 \\ K_{II} = 3 \end{matrix}}$$

Energy release rate of J integral: Assumptions

1. Homogeneous body
2. Linear or non-linear elastic solid   no plasticity
3. No inertia, or body forces; no initial stresses $a=0$ $b=0$
4. No thermal loading
5. 2-D stress and deformation field
6. Plane stress or plane strain
7. Mode I loading
8. Stress free crack

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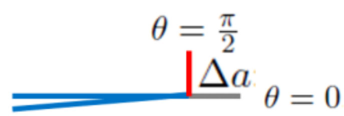
We'll see a more general form of J integral removes ALL these restrictions!

J-K relationship

In fact both J1 (J) and J2 are related to SIFs:

$$J_1 = \int_\Gamma \left(w dy - \mathbf{t}\frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)$$
$$J_2 = \int_\Gamma \left(-w dx - \mathbf{t}\frac{\partial \mathbf{u}}{\partial y} d\Gamma \right)$$

J1 & J2: crack advance for (θ = 0, 90) degrees



$$\begin{aligned} J &= J_1 - iJ_2 \\ &= \frac{(1+\nu)(1+\kappa)}{4E}(K_I^2 + K_{II}^2 + 2iK_I K_{II}) \end{aligned}$$

Hellen and Blackburn (1975)

$$\boxed{\begin{aligned} J_1 &= \frac{K_I^2 + K_{II}^2}{E'} \\ J_2 &= \frac{-2K_I K_{II}}{E'} \end{aligned}}$$

$$E' = \begin{cases} E & \text{plane strain} \\ \frac{E}{1-\nu^2} & \text{plane stress} \end{cases}$$

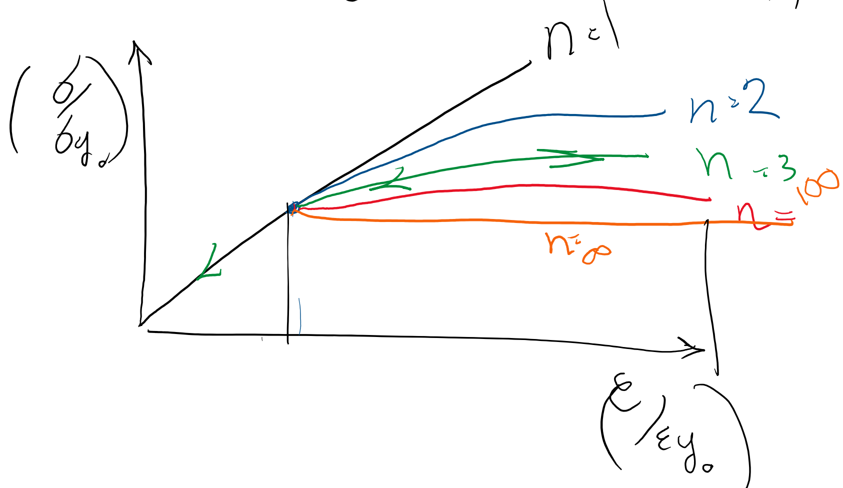
Note that if $K_I = a, K_{II} = b$ is a solution the general solution is:
$K_I = \pm a, K_{II} = \pm b$ and $K_I = \pm b, K_{II} = \pm a$

5.3.5 plastic crack tip fields:
Hutchinson, Rice, Rosengren (HRR) solution

Ramberg-Osgood model (Nonlinear elasticity)

$$\frac{\varepsilon}{\varepsilon_{y_0}} = \underbrace{\frac{\sigma}{\sigma_{y_0}}}_{\text{like } \varepsilon_e} + \underbrace{\alpha\left(\frac{\sigma}{\sigma_{y_0}}\right)^n}_{\text{like } \varepsilon_p}$$

$\varepsilon = \varepsilon_e + \varepsilon_p$ in plasticity



$$\frac{\varepsilon}{\varepsilon_{y_0}} = \cancel{\left(\frac{\sigma}{\sigma_{y_0}}\right)} + \underbrace{\alpha\left(\frac{\sigma}{\sigma_{y_0}}\right)^n}_{\text{this term that is similar to } \varepsilon_p \text{ dominates}}$$

we ignore this

$$\boxed{\varepsilon \propto \sigma^n}$$

close to the crack tip in RO model