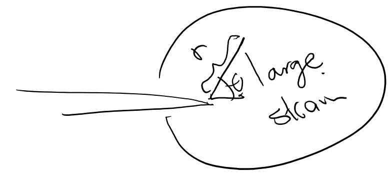
FM20241023

Wednesday, October 23, 2024 9:46 AM

Ramberg-Osgood model

From last time, around the crack tip we had:

$$\varepsilon \propto \sigma^n$$

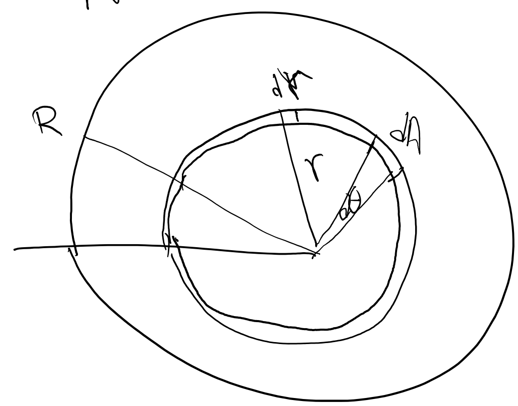


around the crack tip

$$\sigma = \frac{c_1}{r^x} \qquad \varepsilon = \frac{c_2}{r^y}$$

strain energy in circle of radius

$$\mathcal{E}(R) = \int W(\varepsilon)\, dA$$



$$\propto \int_0^{2\pi}\int_0^R \underbrace{\frac{1}{r^x}}_{\text{stress}} \underbrace{\frac{1}{r^y}}_{\text{strain}} \overbrace{(r\,dr\,d\theta)}^{dA}$$

$$= \int_0^{2\pi}\int_0^R r^{\overbrace{1-x-y}^{\text{Power}}}\, dr\, d\theta$$

For the energy to be finite & have certain R dependency we require

$$1 - x - y \geq 0 \qquad x + y \leq 1$$

In fact $\boxed{x + y = 1}$ (i)

(2) $\varepsilon \propto \sigma^n$

$\varepsilon = \frac{c_2}{r^y}$, $\sigma = \frac{c_1}{r^x}$ $\Longrightarrow r^{-y} = r^{-nx} \Rightarrow \boxed{y = nx}$ (ii)

(i),(ii) $\Longrightarrow$ $y = \frac{n}{n+1}$ $\quad \sigma \propto \frac{1}{\ldots}$, $\varepsilon \sim 1\ldots$

$$y = \frac{n}{n+1} \qquad (1) \qquad x = \frac{1}{n+1}$$

$$\sigma \propto \frac{1}{r^{\frac{1}{n+1}}}, \quad \varepsilon \propto \frac{1}{r^{\frac{n}{n+1}}}$$

$n=1$: Linear elastic

$$\sigma, \varepsilon \propto \frac{1}{\sqrt{r}} \quad \text{LEFM}$$

LEFM

$$\sigma_{ij}^{I}(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} f_{ij}^{I}(\theta)$$

R.O

$$\frac{\varepsilon}{\varepsilon_{y_0}} = \left(\frac{\sigma}{\sigma_{y_0}}\right) + \alpha\left(\frac{\sigma}{\sigma_{y_0}}\right)^n$$

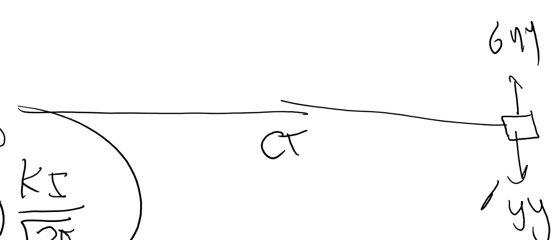
$n \to \infty$ σ bounded!

$$\varepsilon \propto \frac{1}{r}$$ all singularity in strain

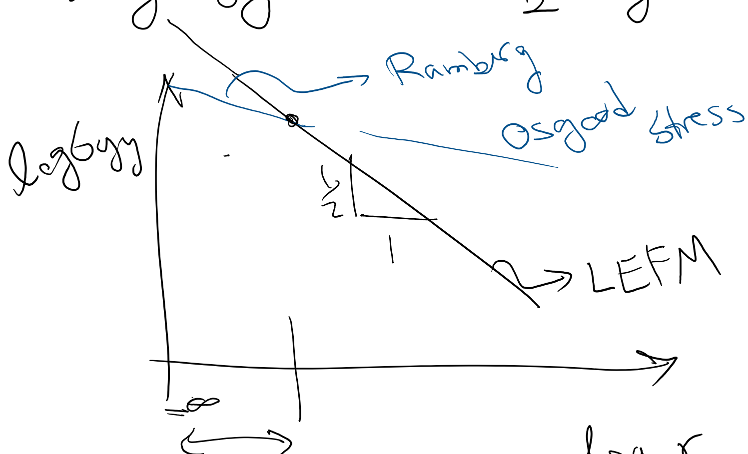
$1 < n < \infty$ ε is always more singular

LEFM

$$\sigma_{yy}(r,\theta=0) = \frac{K_I}{\sqrt{2\pi r}}$$



$$\log \sigma_{yy} = -\frac{1}{2}\log r + \underbrace{\log \frac{K_I}{\sqrt{2\pi}}}_{A}$$

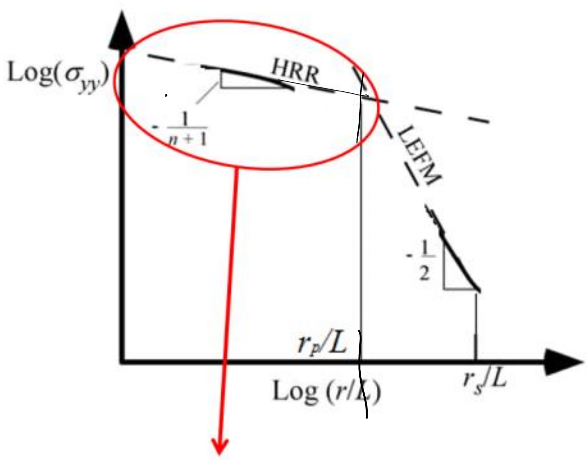


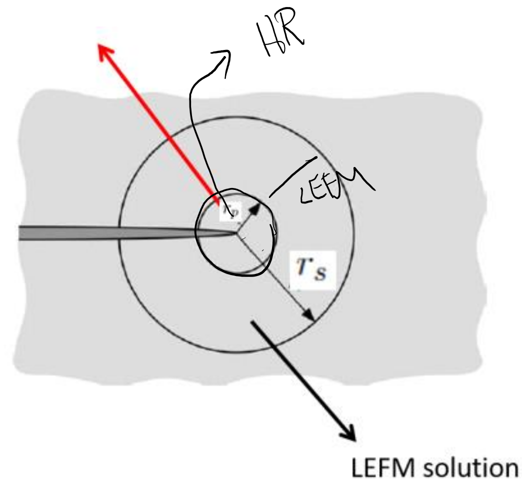
Ramberg-Osgood

$$\sigma_{yy} \propto \frac{1}{r^{\frac{1}{1+n}}}$$

$$\Downarrow$$

$$\log \sigma_{yy} = -\frac{1}{1+n}\log r + \underbrace{C}_{\text{constant}}$$





Stress is still singular but with a weaker power of singularity!

$$\sigma \propto \frac{1}{r^{\frac{1}{n+1}}} \qquad \varepsilon \propto \frac{1}{r^{\frac{n}{n+1}}}$$

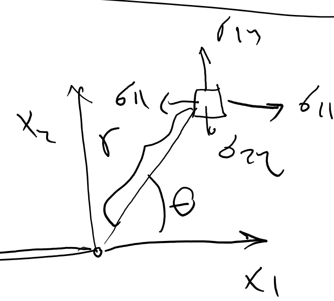
What about the full expression of $\sigma(r,\theta)$

plays the role of K

$$\sigma_{ij} = \sigma_0 \left(\frac{EJ}{\alpha\sigma_0^2 I_n r}\right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(n,\theta)$$

$$\epsilon_{ij} = \frac{\alpha\sigma_0}{E}\left(\frac{EJ}{\alpha\sigma_0^2 I_n r}\right)^{\frac{n}{n+1}} \tilde{\epsilon}_{ij}(n,\theta)$$

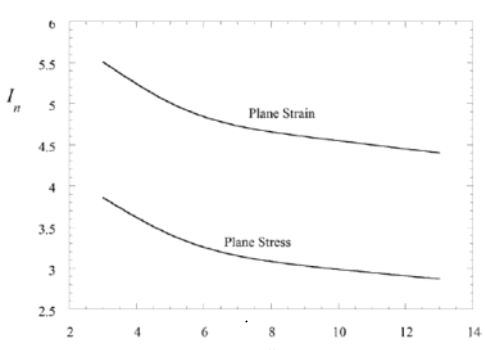
generalization of LEFM



$$\sigma^I_{ij}(r,\theta) = \frac{K_I}{2\pi r^{1/2}} f^I_{ij}(\theta)$$

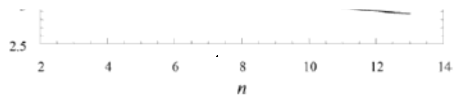
RO

$$\left(\frac{\varepsilon}{\varepsilon_{y_0}}\right) = \left(\frac{\sigma}{\sigma_0}\right) + \alpha\left(\frac{\sigma}{\sigma_0}\right)^n$$

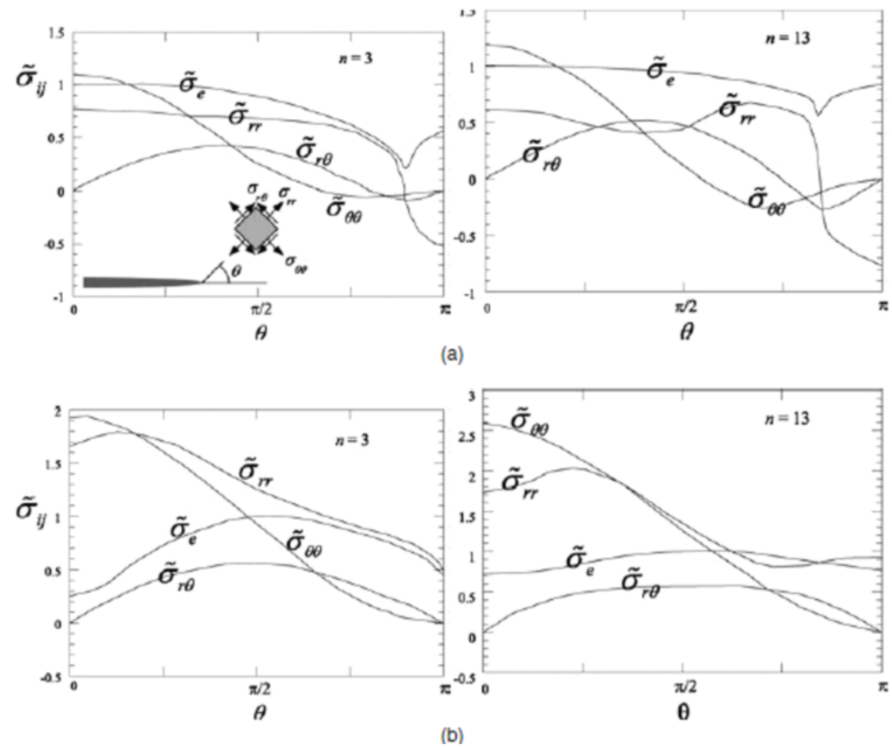


Dim Analysis of $\left[\frac{EJ}{\alpha\sigma_0^2 I_n r}\right]$

$$= \frac{[\sigma]\overbrace{[\sigma][L]}^{[J]}}{[\sigma]^2 \underbrace{[L]}_{[r]}} = 1$$



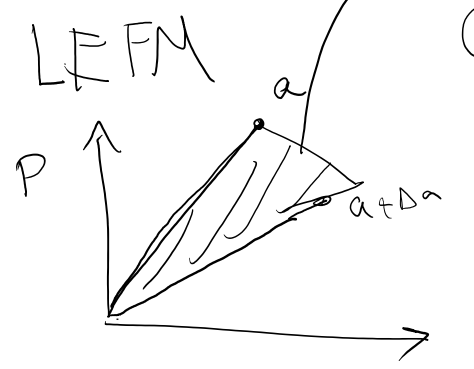
HRR solution:Angular functions

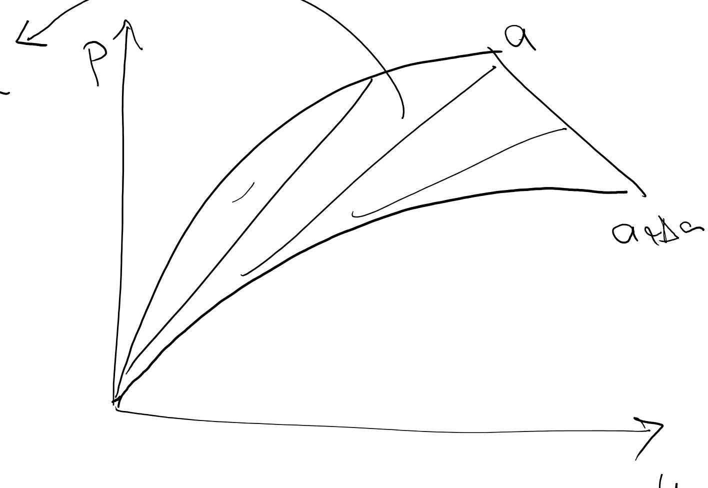


5.3. 4. Energy Release Rate, crack growth and R curves

LEFM

$G = \frac{\text{shaded area}}{B\Delta a}$

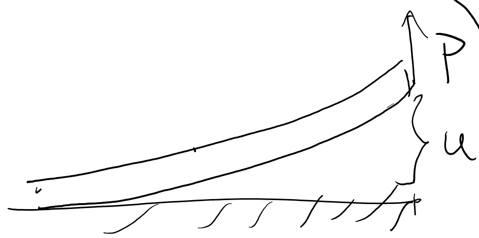


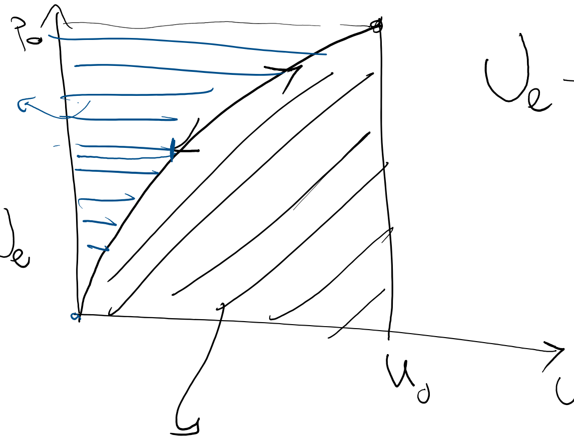


Nonlinear elasticity

Complementary internal energy

$U_e^* = \int_0^{P_0} u(P)\,dP = u_0 P_0 - U_e$



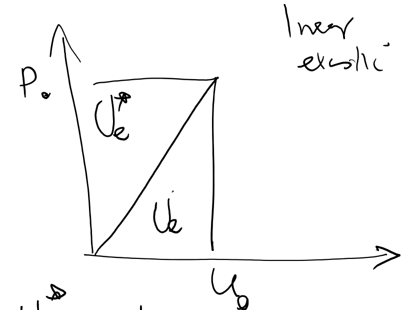


$U_e + U_e^* = P_0 u_0$

We're going to use U_e & U_e^* to express G

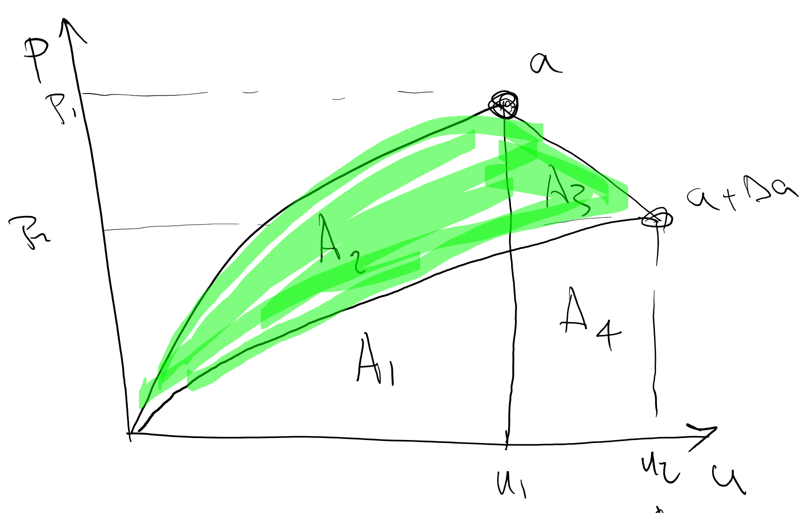
Internal energy $U_e = \int_{u=0}^{u=u_0} P(u)\,du$

linear elastic



$U_e = U_e^* = \frac{1}{2} P_0 u_0$

Why $G = \frac{\text{shaded area}}{B \Delta a}$



$G = -\frac{\Delta \Pi}{\Delta A} = -\frac{\Delta(U_e - W_{ext})}{\Delta B a}$

$\lim_{\Delta A \to 0}$

$G = \frac{1}{B}\frac{1}{\Delta a}\left(-U_{e_2} + U_{e_1} - W_{ext_{1-2}}\right)$

$U_{e_1} = A_1 + A_2$

$U_{e_2} = A_1 + A_4$

$W_{ext_{1\to 2}} = \int_{u_1}^{u_2} P\,du = A_3 + A_4$

shaded area

$U_{e_2} = A_1 + A_4$ | $W_{ext_{1\to2}} = \int_{u_1} = A_3 + A_4$

shaded area

$$G = \frac{1}{B\Delta a}\left(-(A_1+A_4) + (A_1+A_2) + (A_3+A_4)\right) = \frac{A_2 + A_3}{B\Delta a}$$

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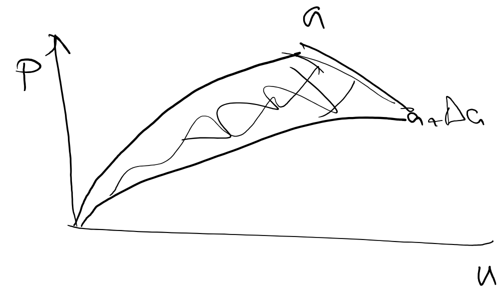
(2) Similar to linear elastic case we have

$$G = \frac{\text{shaded area}}{B\Delta a}$$



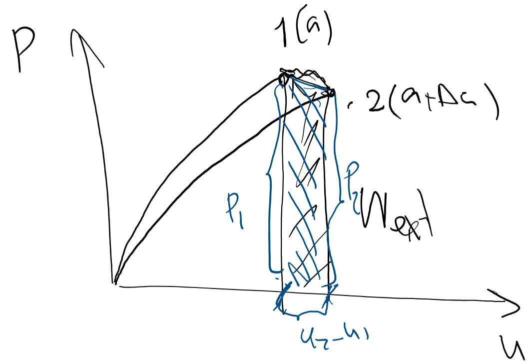
Can we have a relation like $\underbrace{G = \frac{P^2}{2B}\frac{dC}{da} = \frac{-u^2}{2B}\frac{dk}{da}}_{\text{linear elasticity}}$

but for nonlinear elasticity

$$G = \lim_{\Delta a \to 0} \frac{-1}{B\Delta a}\left(\Delta\Pi_e\right) = \lim_{\Delta a\to 0}\frac{1}{B\Delta a}\left(-\Delta U_e + W_{ext_{1\to2}}\right)$$

$$= \frac{1}{B}\left\{\left(-\lim_{\Delta a\to 0}\frac{\Delta U_e}{\Delta a}\right) + \lim_{\Delta a\to 0}\frac{\int_{u_1}^{u_2} P\,du}{\Delta a}\right\}$$

$\approx \left(\frac{P_1+P_2}{2}\right)(u_2-u_1)$



$$= \frac{1}{B}\left\{-\lim_{\Delta a\to 0}\frac{\Delta U_e}{\Delta a} + \lim_{\Delta a\to 0}\left(\frac{P_1+P_2}{2}\right)\frac{(u_2-u_1)}{\Delta a}\right\}$$

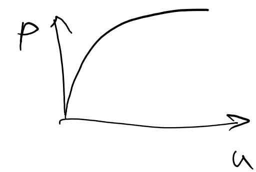
$\to \frac{u(a+\Delta a) - u(a)}{\Delta a}$

$$G = \frac{1}{B}\left\{-\frac{dU_e(a)}{da} + P\frac{du}{da}\right\}$$

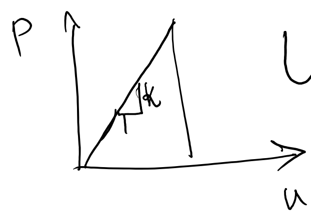
Differential formula

Nonlinear elasticity

$$\boxed{G = \frac{1}{B}\left\{ -\frac{dU_e}{da} + P\frac{du}{da}\right\}} \quad (3)$$



What if the material is linear



$$U_e = \frac{1}{2}Pu = \frac{1}{2}ku^2$$

plug into (3)

$$G = \frac{1}{B}\left\{ -\frac{d\overbrace{\tfrac{1}{2}ku^2}^{U_e}}{da} + P\frac{du}{da}\right\} =$$

$$\frac{1}{B}\left\{ -\frac{1}{2}\frac{dk}{da}u^2 + \frac{-k}{2}\times 2u\frac{du}{da} + P\frac{du}{da}\right\} =$$

$k \cdot u = P$

$$\frac{-u^2}{2B}\frac{dk}{da} + \frac{1}{B}\left(-P\frac{du}{da} + P\frac{du}{da}\right) = \frac{-u^2}{2B}\frac{dk}{da}$$

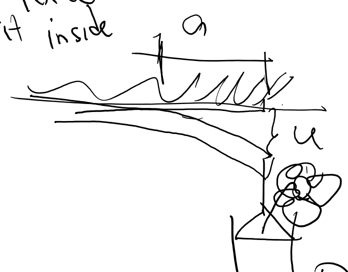
we recovered the simpler linear relation we had before

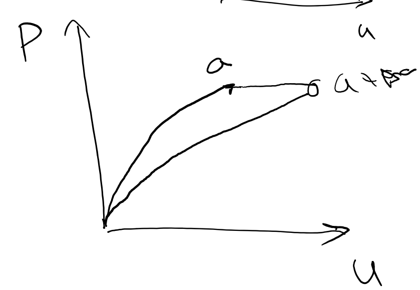
Special cases of nonlinear elasticity G formula:

Case 1: dead load

$$\boxed{G = \frac{1}{B}\left\{ -\frac{dU_e}{da} + P\frac{du}{da}\right\}} \quad (3)$$

since P is fixed we can move it inside $\frac{d}{da}$



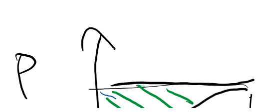


$$G = \frac{1}{B}\left\{ -\frac{dU_e}{da} + P\frac{du}{da}\right\}$$

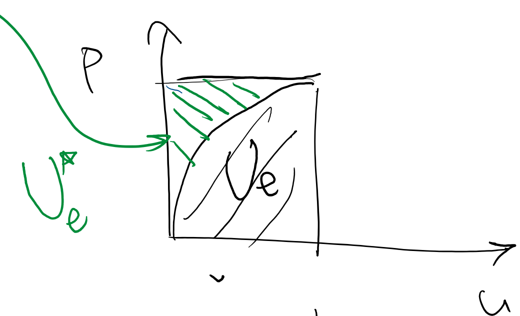
$$G = \frac{1}{B}\left\{ -\frac{dU_e}{da} + \frac{dPu}{da}\right\}$$

P is fixed

$$\frac{1}{B}\left\{ \frac{d}{}(Pu - U_e)\right.$$

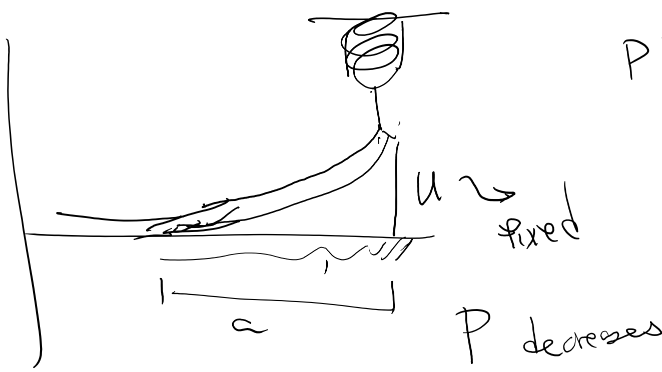


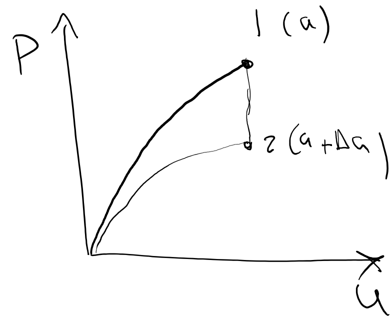
$$= \frac{1}{B}\left\{\frac{d(Pu - U_e)}{da}\right.$$



Dead load $\boxed{G = \frac{1}{B}\frac{dU_e^*}{da}}$

$$G = \frac{1}{B}\left(\frac{-dU_e}{da} + P\cancel{\frac{du}{da}}\right) \to 0$$





Fixed grip $\boxed{G = \frac{-1}{B}\frac{dU_e}{da}}$

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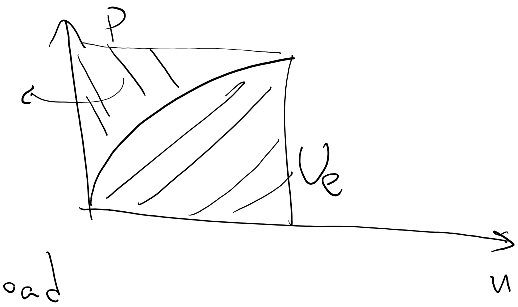
(4)

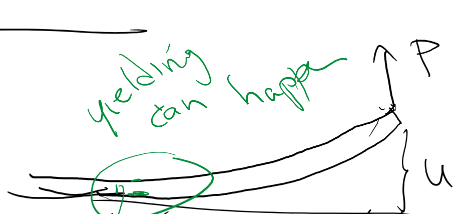
Summary

$$G = \frac{1}{B}\left(-\frac{dU_e}{da} + P\frac{du}{da}\right) \text{ (general)}$$

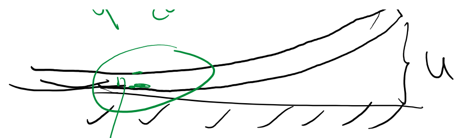


$$= \frac{1}{B}\frac{dU_e^*}{da} \quad \text{dead (fixed) load}$$

$$= -\frac{1}{B}\frac{dU_e}{da} \quad \text{fixed grip}$$



→ dead load $G = \frac{1}{B}\frac{dU_e^*}{da}$



$u = \frac{\;}{B} \; \frac{\;}{da}$