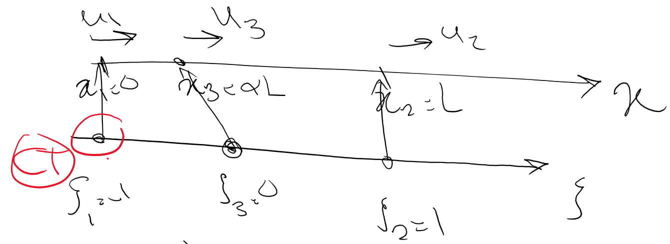
FM20241106

Wednesday, November 6, 2024 9:48 AM

From last time:

$\varepsilon \to 0$ at CT

$\xi \to -1$



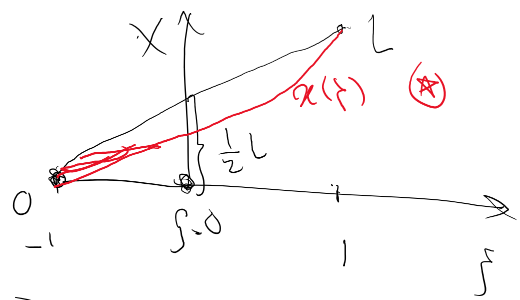
(1) $u(\xi) = u_1 N_1(\xi) + u_2 N_2(\xi) + u_3 N_3(\xi)$

$$x(\xi) = \underset{0}{x_1} N_1(\xi) + \underset{L}{x_2} N_2(\xi) + \underset{\alpha L}{x_3} N_3(\xi) \Rightarrow (\star) \quad \boxed{x(\xi) = L\left(\alpha + \frac{\xi}{2} + \xi^2\left(\frac{1}{2} - \alpha\right)\right)}$$

$$\varepsilon(x) = \frac{du}{dx} = \frac{\frac{du}{d\xi}}{\frac{dx}{d\xi}}$$

$\frac{du}{d\xi}$ → computed from 1

$\frac{dx}{d\xi}$ computed from $(\star)$



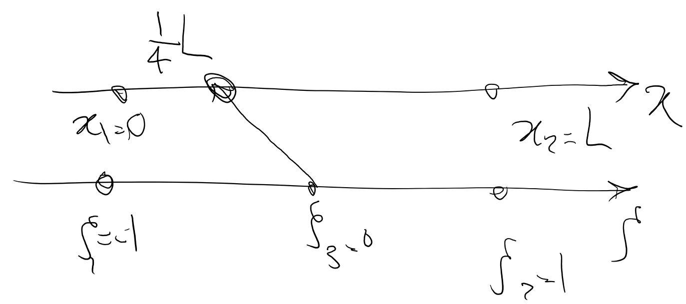
to creat $\varepsilon \to \infty$ @ $\xi = -1$

denominator must be zero @ $\xi = -1$

$$(\star) \Rightarrow \boxed{\frac{dx}{d\xi} = L\left(\frac{1}{2} + 2\xi\left(\frac{1}{2} - \alpha\right)\right)} \quad (\star\star)$$

$$(\star\star) \text{ for } \xi = -1 \Rightarrow \frac{dx}{d\xi}(\xi=-1) = L\left(\frac{1}{2} + 2(-1)\left(\frac{1}{2} - \alpha\right)\right) = 0$$

$$\Rightarrow \boxed{\alpha = \frac{1}{4}}$$



for $\alpha = \frac{1}{4}$ from $(\star)$

$$\boxed{x = \frac{L}{4}(\xi+1)^2 \Rightarrow \xi = 2\sqrt{\frac{x}{L}} - 1} \quad (2)$$

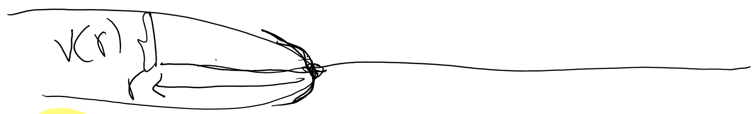
eqn 1 $\quad u(\xi) = \sum_{i=1}^{3} u_i N_i(\xi) = u_1 \frac{\xi(\xi-1)}{2} + u_2 \frac{\xi(\xi+1)}{2} + u_3(1-\xi^2)$

$$\boxed{u = u_1 + \sqrt{\frac{x}{L}}\left(-3u_1 - u_2 + 4u_3\right) + \frac{2x}{L}\left(u_1 + u_2 - 2u_3\right)}$$

comparison

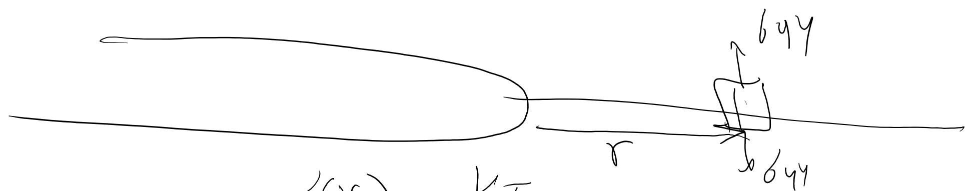
$$u = u_1 + \frac{\sqrt{x}}{\sqrt{L}}\left(-3u_1 - u_2 + 4u_3\right) + \frac{2x}{L}\left(u_1 + u_2 - 2u_3\right)$$

comparison to CT displacement

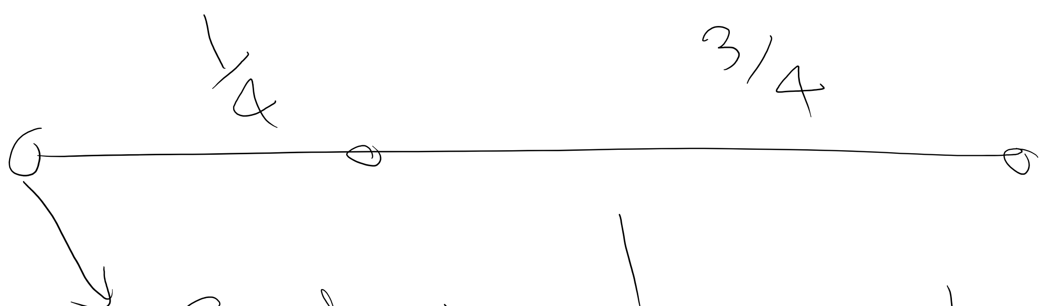


$$V(r) = \frac{(\kappa+1)K_I}{\sqrt{2\pi}\,\mu}\sqrt{r}$$

$$\varepsilon(x) = \frac{du}{dx} = \frac{1}{\sqrt{x}\sqrt{L}}\left(-\frac{3}{2}u_1 - \frac{u_2}{2} + 2u_3\right) + \frac{2}{L}\left(u_1 + u_2 - 2u_3\right)$$



$$\sigma_{yy}(r) = \frac{K_I}{\sqrt{2\pi r}}$$

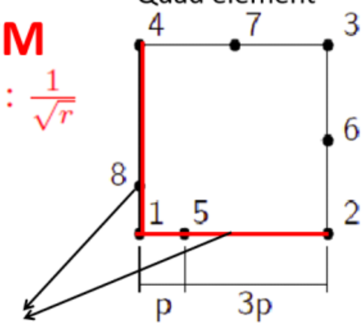


creates $\frac{1}{\sqrt{r}}$ stress/strain singularity!

Isoparametric singular elements

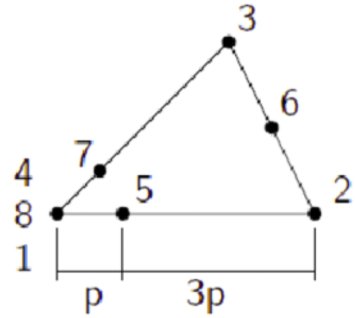
- **LEFM**
  $\epsilon, \sigma : \frac{1}{\sqrt{r}}$

Quarter point
Quad element



singular form $\frac{1}{\sqrt{r}}$ only along these lines
NOT recommended

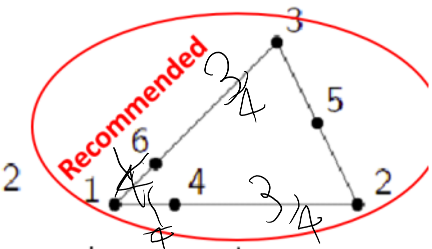
Quarter point **collapsed**
Quad element



Improvement:
- $\frac{1}{\sqrt{r}}$ from inside all element

Problem
- Solution inaccuracy and sensitivity when opposite edge 3-6-2 is curved

Quarter point
Tri element



Improvement:
- Better accuracy and less mesh sensitivity

How should the finite element mesh look like around a CT?

Strain singularity at ξ means

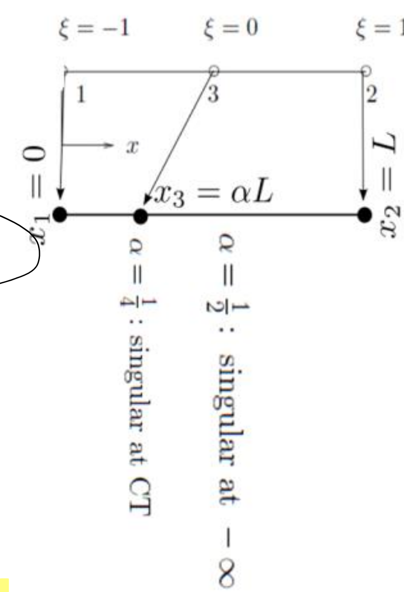
$$\frac{dx}{d\xi} = L\left\{2\xi(\frac{1}{2} - \alpha) + \frac{1}{2}\right\} < 0 \qquad \xi < -1$$

must be zero. Accordingly,

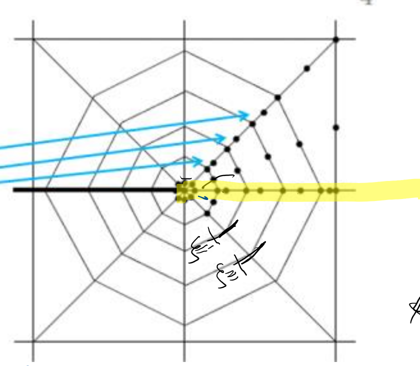
Singularity at infinity($\xi \to -\infty$) $\qquad \alpha -> (\frac{1}{2})^-$

Singularity at crack tip($\xi = -1$) $\qquad \alpha = \frac{1}{4}$

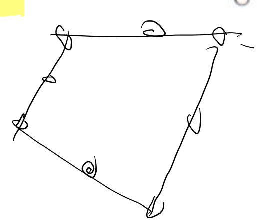
Singularity inside element(not of interest)($-1 < \xi < 0$) $\qquad 0 < \alpha < \frac{1}{4}$



- Transition elements:
  According to this analysis
  mid nodes of next layers move to ½ point from ¼ point



Lynn and Ingraffea 1977)

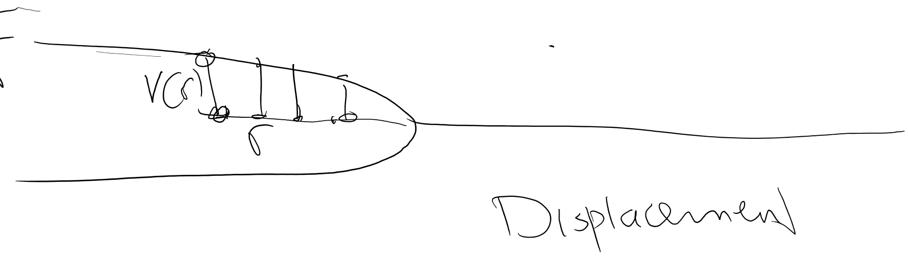


We'll create meshes like this next week

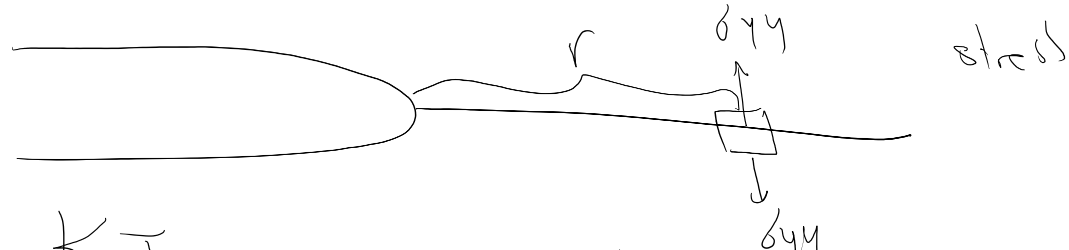
Now that we have the FEM mesh around the crack tip, how do we calculate K and G?

6.1.3. Extraction of K (SIF), G

$$v(r) \propto K_I \sqrt{r}$$



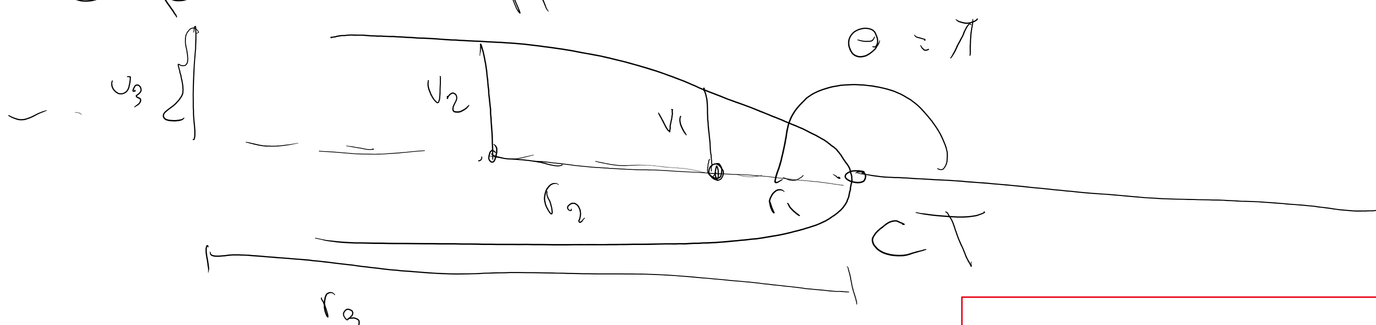
$$K_I \propto \frac{v(r)}{\sqrt{r}}$$



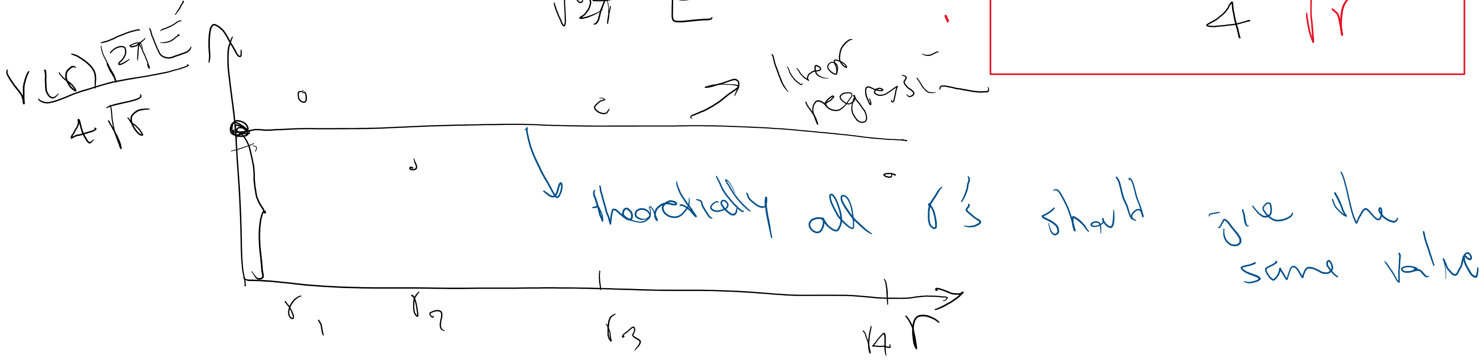
$$\sigma_{yy}(r) = \frac{K_I}{\sqrt{2\pi r}} \qquad \underline{K_I = \sigma_{yy}\sqrt{2\pi r}}$$

very poor computationally

Displacement approach

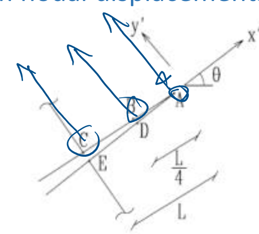


$$v(r, \theta = \pi) = \frac{4 K_I \sqrt{r}}{\sqrt{2\pi}\, E'} \Rightarrow \text{③} \quad \boxed{K_I = \frac{v(r)\sqrt{2\pi}\, E'}{4\sqrt{r}}}$$



- With special CT elements (1/4, 3/4 mid-node location), we can also calculate K from nodal displacements

or alternatively from the first quarter point element:



$$v = K_I \frac{\kappa+1}{2G}\sqrt{\frac{r}{2\pi}}$$

$$u' = \overline{u}'_A + \left(-3\overline{u}'_A + 4\overline{u}'_B - \overline{u}'_C\right)\sqrt{\frac{r}{L}} + \left(2\overline{u}'_A + 2\overline{u}'_C - 4\overline{u}'_B\right)\frac{r}{L}$$

$$v' = \overline{v}'_A + \left(-3\overline{v}'_A + 4\overline{v}'_B - \overline{v}'_C\right)\sqrt{\frac{r}{L}} + \left(2\overline{v}'_A + 2\overline{v}'_C - 4\overline{v}'_B\right)\frac{r}{L}$$

$$\boxed{K_I = \frac{2G}{\kappa+1}\sqrt{\frac{2\pi}{L}}\left(-3\overline{v}'_A + 4\overline{v}'_B - \overline{v}'_C\right)}$$

Recall for 1D

$$u = u_1 + \frac{\sqrt{x}}{\sqrt{L}}(-3u_1 - u_2 + 4u_3) + \frac{2x}{L}(u_1 + u_2 - 2u_3)$$

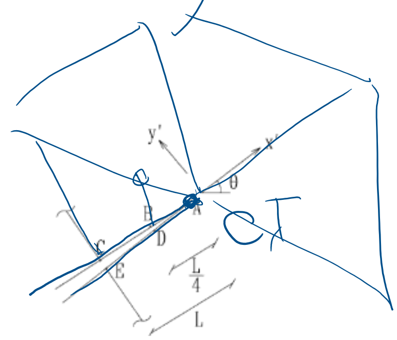
$$K_I \frac{\kappa+1}{2G}\sqrt{\frac{r}{2\pi}} = \left(-3\overline{v}'_A + 4\overline{v}'_B - \overline{v}'_C\right)\sqrt{\frac{r}{L}}$$

$$\Longrightarrow \boxed{K_I = \frac{2G}{\kappa+1}\sqrt{\frac{2\pi}{L}}\left(-3\overline{v}'_A + 4\overline{v}'_B - \overline{v}'_C\right)}$$

pure mode I fracture

What about mixed mode fracture? Can we used behind the crack displacements to extract KI and KII?

first layer of elements



$$\begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} = \frac{1}{2}\frac{2G}{\kappa+1}\sqrt{\frac{2\pi}{L}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -3\overline{u}'_A + 4\left(\overline{u}'_B - \overline{u}'_D\right) - \left(\overline{u}'_C - \overline{u}'_E\right) \\ -3\overline{v}'_A + 4\left(\overline{v}'_B - \overline{v}'_D\right) - \left(\overline{v}'_C - \overline{v}'_E\right) \end{bmatrix}$$

Mixed mode generalization:

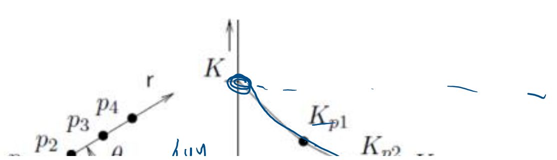
we'll do this next week too

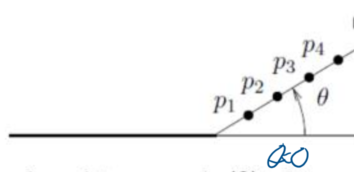
K from the stress field ahead of the crack

2. Stress

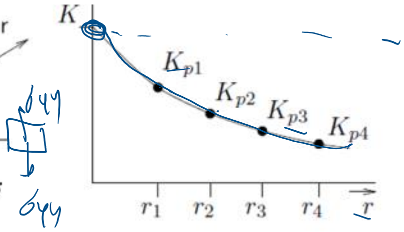
$$\sigma_{yy}(r, \theta=0) = \frac{K_I}{\sqrt{2\pi r}} \Rightarrow \boxed{K_I = \sqrt{2\pi r}\,\sigma_{yy}(r)}$$

$$\boxed{K_I = \lim_{r\longrightarrow 0}\left(\sqrt{2\pi r}\;\sigma_{22}|_{\theta=0}\right) \quad ; \quad K_{II} = \lim_{r\longrightarrow 0}\left(\sqrt{2\pi r}\;\sigma_{12}|_{\theta=0}\right)}$$





or can be done for arbitrary angle (θ) taking σ angular dependence *f*(θ) into account



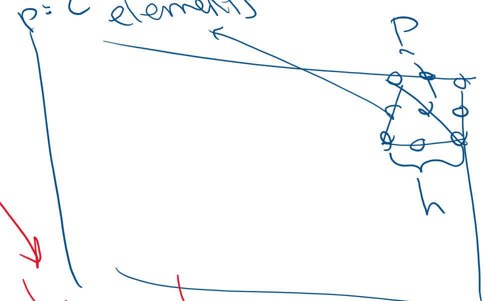
Computationally, it's much better to use the displacement approach for the following reasons:

1. $\sigma \propto \frac{1}{\sqrt{r}}$ as $r \to 0$

$V \propto \sqrt{r}$

much more well-behaved and easier to capture numerically

2. $\|u - u_{exact}\|$ error in displacement $\propto C h^{p+1}$



dropped

$\|\varepsilon - \varepsilon_{exact}\| \propto C' h^{p}$

$\varepsilon = \frac{\nabla u + \nabla u^{T}}{2}$

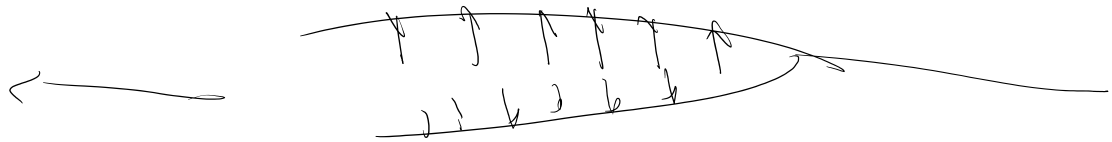
1 der

Strain & stress are less accurate in FEM in general

3. stress approach is sensitive to tractions

?. Stress approach is sensitive to ... stress

on the crack surface



stress approach

is even worse

Other approaches indirectly calculate K from G (or J)

2. K from energy approaches

1. Elementary crack advance (two FEM solutions for a and $a + \Delta a$)
2. Virtual Crack Extension: Stiffness derivative approach
3. J-integral based approaches (next section)

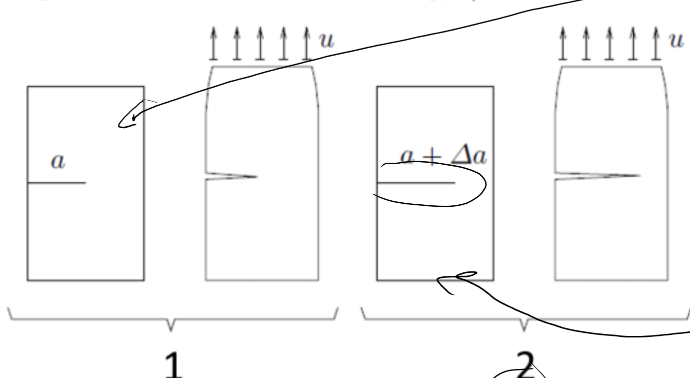
After obtaining *G* (or *J*=*G* for LEFM) K can be obtained from

$$K_I^2 = E'G \quad \checkmark$$

$$E' = \begin{cases} E & \text{plane stress} \\ \frac{E}{1-\nu^2} & \text{plane strain} \end{cases}$$

2.1 Elementary crack advance

For fixed grip boundary condition perform two simulations (1, *a*) and (2, *a*+Δ*a*):
All FEM packages can compute strain (internal) energy U_i



$$G = -\frac{d\Pi_e}{dA} = -\frac{1}{B}\frac{d\Pi_e}{da} \approx -\frac{1}{B}\left(\frac{\Pi_e(a+\Delta a)-\Pi(a)}{\Delta a}\right)$$

$$G = -\frac{d\Pi_e}{dA} = -\frac{1}{B}\left(\frac{d\Pi_e}{da}\right) \approx -\frac{1}{B}\left(\frac{\Pi_e(a+\Delta a) - \Pi(a)}{\Delta a}\right)$$

$\Delta a \to 0$

1. Delta a small -> bad run into finite precision errors
2. Delta a large -> derivative approximation is getting worse

Other pitfall is that we need two FEM solutions

2.2 Virtual crack extension

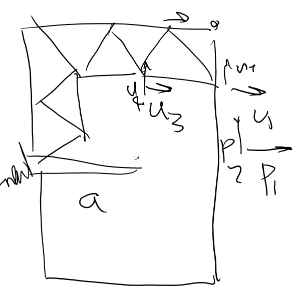
$$G = -\frac{d\Pi_e}{dA} = -\frac{1}{B}\frac{d\Pi_e}{da}$$

$$\Pi_e = U_e - W \rightarrow \text{external work}$$

in FEM $\quad U_e = \frac{1}{2}u^t K u$

$K \rightarrow$ stiffness matrix

$u \rightarrow$ nodal displacement vector



$$W = Pu$$

$$G = -\frac{1}{B}\frac{d(U_e - W)}{da} = -\frac{1}{B}\frac{d}{da}\left(\frac{1}{2}u^t K u - Pu\right)$$

equal

$$G = -\frac{1}{B}\left(\frac{1}{2}\left(\frac{du}{da}\right)^t K u + \underbrace{\frac{1}{2}u^t \frac{dK}{da} u}_{} + \frac{1}{2}u^t K \frac{du}{da} - \frac{dP}{da}u - P\frac{du}{da}\right)$$

$$\left(\frac{du}{da}\right)^t K u = \left(\left(\frac{du}{da}\right)^t K u\right)^t = u^t K^t \frac{du}{da}$$

in FEM $K^t = K$ $\quad = u^t K \frac{du}{da}$

in u $K-P$ $\qquad = u \; K \frac{dd}{de}$

$$BG = \left(-\frac{du}{da} \underbrace{Ku}_{P} + P\frac{du}{da} \right) - \frac{1}{2} u^t \frac{dK}{da} u + \frac{dP}{da} u$$

FEM $\qquad Ku = P$

$$\boxed{\begin{gathered} BG = -\frac{1}{2} u^t \frac{dK}{da} u + \frac{dP}{da} u \\ \text{in many problems} \\ \text{loads do not depend on } a \\ \frac{dP}{da} = 0 \\ \Rightarrow \text{ most cases } \quad BG = -\frac{1}{2} u^t \frac{dK}{da} u \end{gathered}}$$