FM20241111

Monday, November 11, 2024   9:43 AM

From last time

2.2 Virtual crack extension

- Potential energy is given by

$$\Pi = \frac{1}{2}\lfloor u \rfloor [K]\{u\} - \lfloor u \rfloor \{p\}$$

$$-G = \frac{\partial \Pi}{\partial a} = \frac{\partial \lfloor u \rfloor}{\partial a}[K]\{u\} + \frac{1}{2}\lfloor u \rfloor \frac{\partial [K]}{\partial a}\{u\} - \frac{\partial \lfloor u \rfloor}{\partial a}\{P\} - \lfloor u \rfloor \frac{\partial \{P\}}{\partial a}$$

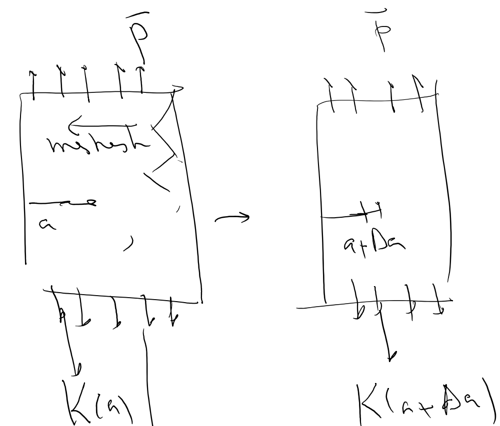
$$= -\frac{\partial \lfloor u \rfloor}{\partial a}\underbrace{([K]\{u\} - \{P\})}_{0} + \frac{1}{2}\lfloor u \rfloor \frac{\partial [K]}{\partial a}\{u\} - \lfloor u \rfloor \frac{\partial \{P\}}{\partial a}$$

$$\boxed{G = -\frac{1}{2}\lfloor u \rfloor \frac{\partial [K]}{\partial a}\{u\} + \lfloor u \rfloor \frac{\partial \{P\}}{\partial a}}$$

applied loads are often independent of crack length $\Longrightarrow \frac{\partial P}{\partial a} = 0$

in these cases → FEM stiffness matrix

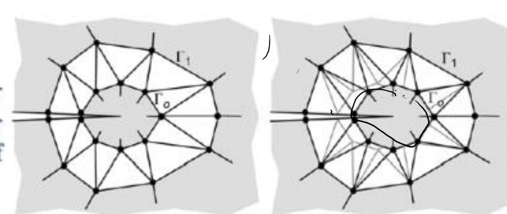
$G = -\frac{1}{2} u \frac{\partial K}{\partial a} u$ → nodal displacement



$$\frac{\partial K}{\partial a} \approx \underbrace{\frac{K(a+\Delta a) - K(a)}{\Delta a}}_{\text{prone to FD errors}}$$

$Ku = P \xRightarrow{\text{solve}} u = K^{-1}P$

- Only the few elements that are distrorted contribute to $\frac{\partial K}{\partial a}$
- We may not even need to form elements and assemble K for a and $a + \Delta a$ to obtain $\frac{\partial K}{\partial a}$. We can explicitly obtain $\frac{\partial k^e}{\partial a}$ for elements affected by crack growth by computing derivatives of actual geometry of the element to parent geometry.



We can analytically see $\frac{dK}{da}$ is with only white element coordinates changing idea

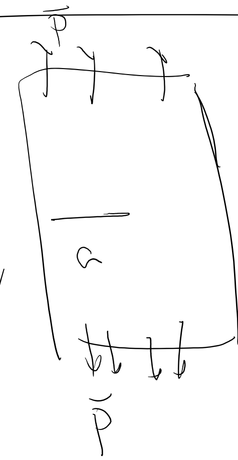
$$K^{e_1} = \frac{AE}{a}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \qquad K^{e_2} = \frac{AE}{L-a}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$\frac{dK^{e_1}}{da} = -\frac{AE}{a^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

J Solution

$Ku = P$

$u = K^{-1}P$ ✓



$\frac{dK}{da}$ is assembled from local $\frac{dK^e}{da}$

&

J: meshing

Perturbed method

- This method is equivalent to J integral method (Park 1974)

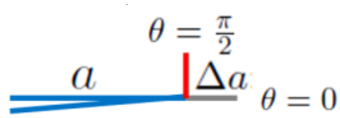
2.2 Virtual crack extension: Mixed mode

- For LEFM energy release rates G_1 and G_2 are given by

$$J_1 = G_1 = \frac{K_I^2 + K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$
$$J_2 = G_2 = \frac{-2K_I K_{II}}{E'}$$

- Using Virtual crack extension (or elementary crack advance) compute G_1 and G_2 for crack lengths a, $a + \Delta a$

$$J_1 = G_1 = \frac{K_I^2 + K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$
$$J_2 = G_2 = \frac{-2K_I K_{II}}{E'}$$



- Obtain K_I and K_{II} from:

$$\boxed{K_I = \frac{s \pm \sqrt{s^2 + \frac{8G_2}{\alpha}}}{4} \qquad K_{II} = \frac{s \mp \sqrt{s^2 + \frac{8G_2}{\alpha}}}{4}}$$

Note that there are two sets of solutions!

$s = 2\sqrt{\frac{G_1 - G_2}{\alpha}}$ and $\alpha = \frac{(1+\nu)(1+\kappa)}{E}$

6.1.4. J integral

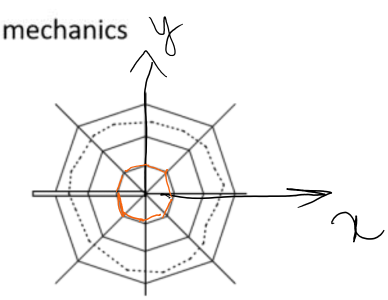
J integral

Uses of J integral:

1. LEFM: Can obtain K_I and K_{II} from J integrals (G = J for LEFM)

$$J_1 = G_1 = \frac{K_I^2 + K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$
$$J_2 = G_2 = \frac{-2K_I K_{II}}{E'}$$

2. Still valid for nonlinear (NLFM) and plastic (PFM) fracture mechanics



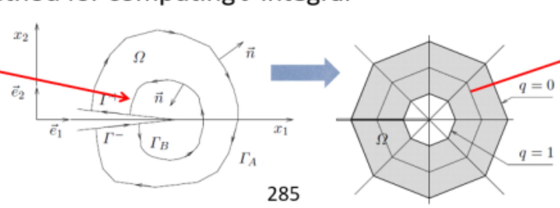
Methods to evaluate J integral:

1. Contour integral:

$$J_1 = \int_\Gamma \left(w dy - \mathbf{t}\frac{\partial \mathbf{u}}{\partial x} d\Gamma \right)$$
$$J_2 = \int_\Gamma \left(-w dx - \mathbf{t}\frac{\partial \mathbf{u}}{\partial y} d\Gamma \right)$$

2. Equlvalent (Energy) **domain** integral (**EDI**):
   - Gauss theorem: line/surface (2D/3D) integral surface/volume integral
   - Much simpler to evaluate computationally
   - Easy to incorporate plasticity, crack surface tractions, thermal strains, *etc.*
   - Prevalent method for computing J-integral

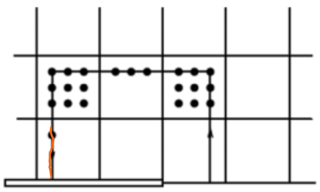
Line integral

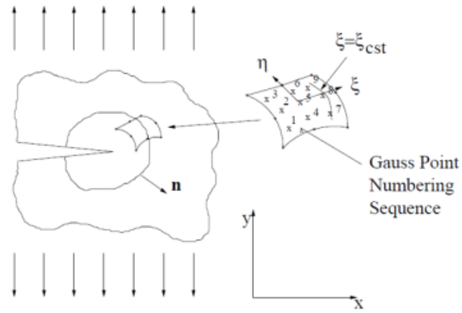


Volume integral

J integral: 1.Contour integral

- Stresses are available and also more accurate at Gauss points
- Integral path goes through Gauss points





$$J = \int_\Gamma w dy - \mathbf{t} \cdot \frac{\partial \mathbf{d}}{\partial x} ds \quad \Rightarrow \quad J^e = \int_{-1}^{1} \left\{ \underbrace{\frac{1}{2}\left[\sigma_x \frac{\partial u}{\partial x} + \tau_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \sigma_y \frac{\partial v}{\partial y}\right]}_{w} \underbrace{\frac{\partial y}{\partial \eta}}_{dy} \right.$$

$$- \underbrace{\left[(\sigma_x n_1 + \tau_{xy} n_2)\frac{\partial u}{\partial x} + (\tau_{xy} n_1 + \sigma_y n_2)\frac{\partial v}{\partial x}\right]}_{\mathbf{t}\cdot\frac{\partial \mathbf{d}}{\partial x}}$$

$$\left. \underbrace{\sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2}}_{ds} \right\} d\eta$$

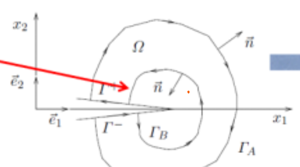
$$= \int_{-1}^{1} I d\eta$$

Cumbersome to formulate the integrand, evaluate normal vector, and integra over lines (2D) and surfaces (3D) $\Rightarrow$

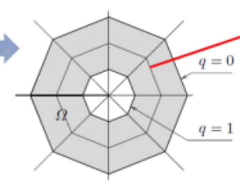
Not commonly used

2. Equlvalent (Energy) **domain** integral (**EDI**):
   - Gauss theorem: line/surface (2D/3D) integral $\Rightarrow$ surface/volume integral
   - Much simpler to evaluate computationally
   - Easy to incorporate plasticity, crack surface tractions, thermal strains, *etc.*
   - Prevalent method for computing J-integral

Line integral



$\Rightarrow$



Volume integral

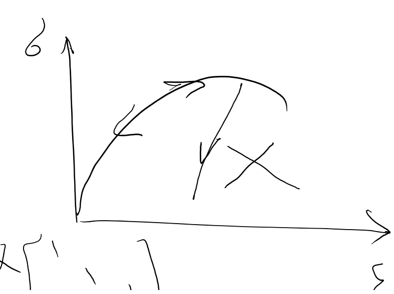
285

Earlier we talked about the limitations of the J integral:

Energy release rate of J integral: Assumptions

1. Homogeneous body
2. Linear or non-linear elastic solid
3. No inertia, or body forces; no initial stresses
   $\rho \ddot{u}$ $\quad b = 0$
4. No thermal loading

$\times$ $\varepsilon_{thermal} = \Delta T \alpha [1, 1,]$



3. No inertia, or body forces, no initial stresses

$\rho\ddot{u}$ $b=0$

4. No thermal loading

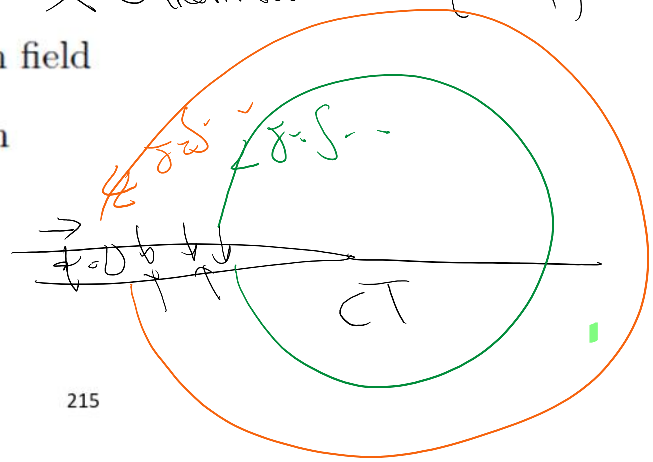
$\times$ $\varepsilon_{thermal} = \Delta t\,\alpha\,[1\ 1\ 1]$ ε

5. 2-D stress and deformation field

6. Plane stress or plane strain

7. Mode I loading

8. Stress free crack



J integral: 2.Equivalent Domain Integral (EDI)

General form of J integral

$$J = \lim_{\Gamma_o \to 0} \int_{\Gamma_o} \left[(w + T)\delta_{1i} - \sigma_{ij} \frac{\partial u_j}{\partial x_1} \right] n_i \, d\Gamma$$

$$w = \int_0^{\varepsilon_{kl}^m} \sigma_{ij} \, d\varepsilon_{ij}^m$$

internal energy density

Inelastic stress

that

the only catch is that $\Gamma_o \to 0$ $\therefore$ the FEM solution is poor here

Kinetic energy density

$$T = \frac{1}{2}\rho \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t}$$

Can include (visco-) plasticity, and thermal stresses

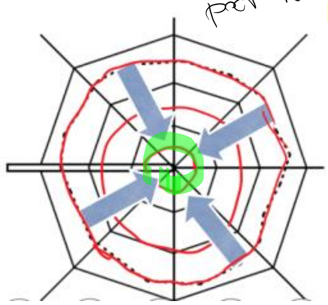
$$\varepsilon_{ij}^{total} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p} + \alpha\Theta\delta_{ij} = \varepsilon_{ij}^{m} + \varepsilon_{kk}^{t}$$

Elastic

Plastic

Can be modeled

Thermal (Θ temperature)



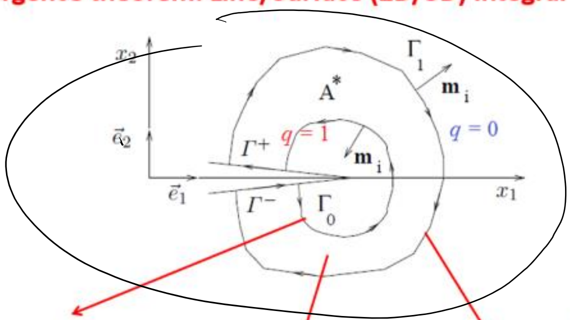
$\Gamma_o \to 0$: J contour approaches Crack tip (CT) $\Rightarrow$

Accuracy of the solution deteriorates at CT $\Rightarrow$

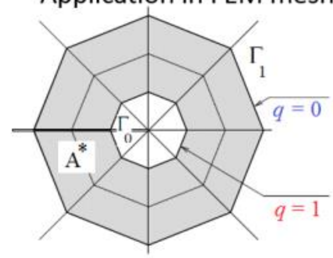
Inaccurate/Impractical evaluation of J using contour integral

J integral: 2. EDI: Derivation

Divergence theorem: Line/Surface (2D/3D) integral $\Rightarrow$ **Surface/Volume Integral**

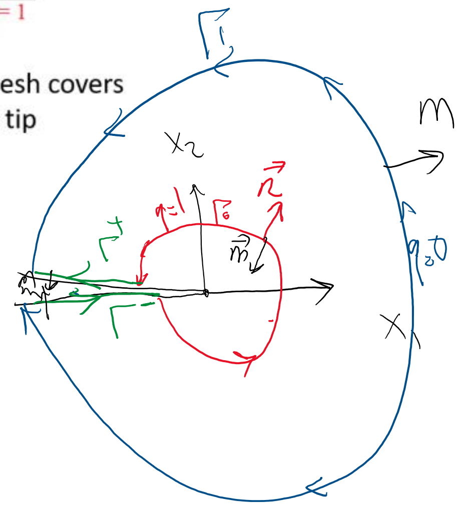


Application in FEM meshes



Original J integral contour

$\Gamma_o \to 0$ 2D mesh covers crack tip



$$J_1 = \int_{\Gamma_0} \underbrace{\left[(W+T)\delta_{i1} - \sigma_{ij}\frac{\partial u_j}{\partial x_1}\right]}_{I\ (\text{integrand})} n_i, \qquad n_i \to -m_i$$

want to take the limit as $\Gamma_0 \to 0$

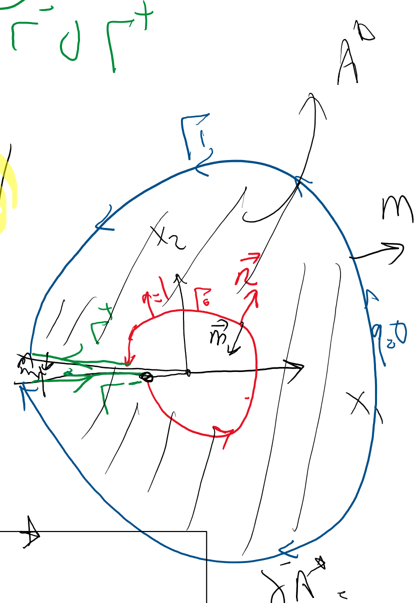
$$-J_1 = \int_{\Gamma_0} I\,(m_i)\,\underbrace{q}_{=1}\,ds + \underbrace{\int_{\Gamma_1} I\, m_i \underbrace{q}_{=0}\, ds}_{0}$$

define q function $q=1$ on Γ_0 & $q=0$ on Γ_1

$$-J_1 + \int_{\Gamma^- \cup \Gamma^+} I m_i q\, ds = \int_{\Gamma_0} I m_i q\, ds + \int_{\Gamma_1} I m_i \underbrace{q}_{0}\, ds + \int_{\Gamma^- \cup \Gamma^+} I m_i q\, ds$$

$$= \int_{\partial A^*} (Iq)\, m_i\, ds = \int_{A^*} \frac{\partial I q}{\partial x_i}\, dV$$

boundary of A^*



$$J_1 = \int_{A^*} \frac{\partial}{\partial x_i}\left[\left(\sigma_{ij}\frac{\partial u_j}{\partial x_1} - (W+T)\delta_{i1}\right) q\right] dV + \int_{\Gamma^- \cup \Gamma^+} \left[(W+T)\delta_{1i} - \sigma_{ij}\frac{\partial u_j}{\partial x_1}\right] m_i q \, ds$$

$\partial A^* = \Gamma_0 \cup \Gamma_1 \cup \Gamma^- \cup \Gamma^+$

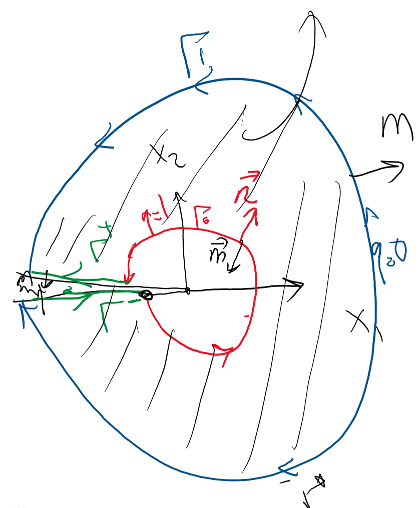
Equivalent domain integral (EDI)

$$J = \int_{\Gamma_o}\left[(w+T)\delta_{1i} - \sigma_{ij}\frac{\partial u_j}{\partial x_1}\right] n_i \, d\Gamma = \int_{\Gamma^*}\left[\sigma_{ij}\frac{\partial u_j}{\partial x_1} - w\delta_{1i}\right] q m_i \, d\Gamma - \int_{\Gamma_+ + \Gamma_-} \sigma_{2j}\frac{\partial u_j}{\partial x_1} q \, d\Gamma$$

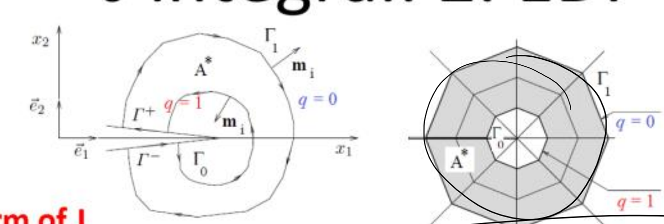
Zero integral on Γ_1 ($q = 0$)

Divergence theorem

$$\boxed{J = \int_{A^*}\frac{\partial}{\partial x_i}\left\{\left[\sigma_{ij}\frac{\partial u_j}{\partial x_1} - w\delta_{1i}\right] q\right\} dA - \int_{\Gamma_+ + \Gamma_-} \sigma_{2j}\frac{\partial u_j}{\partial x_1} q \, d\Gamma}$$



J integral: 2. EDI



General form of J

$$J = \int_{A^*}\left\{\left[\sigma_{ij}\frac{\partial u_j}{\partial x_1} - w\delta_{1i}\right]\frac{\partial q}{\partial x_i} + \left[\sigma_{ij}\frac{\partial \varepsilon^p_{ij}}{\partial x_1} - \frac{\partial w^p}{\partial x_1} + \alpha\sigma_{ii}\frac{\partial \Theta}{\partial x_1} - F_i\frac{\partial u_j}{\partial x_1}\right] q\right\} dA - \int_{\Gamma_+ + \Gamma_-} \sigma_{2j}\frac{\partial u_j}{\partial x_1} q \, d\Gamma$$

Plasticity effects

Thermal effects

Body force

Nonzero crack surface traction

goes away if no plasticity

$= 0$

if goes away
— no plasticity
— thermal strain
— no body force

↓ = 0
if
traction
free
surface

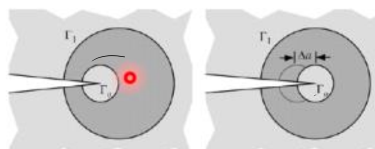
Simplified Case:

(Nonlinear) elastic, no thermal strain, no body force, traction free crack surfaces

$$\boxed{J=\int_{A^*}\left[\sigma_{ij}\frac{\partial u_j}{\partial x_1}-w\delta_{1i}\right]\frac{\partial q}{\partial x_i}dA}$$

This is the same as deLorenzi's approach where $q=\frac{1}{\Delta a}\frac{\partial \Delta x_1}{\partial x_i}$ finds a physical interpretation (virtual crack extension)

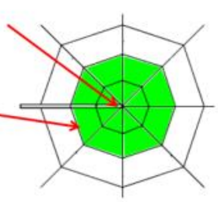
$$\mathcal{G}=\frac{1}{\Delta a}\int_A\left(\sigma_{ij}\frac{\partial u_j}{\partial x_1}-w\delta_{i1}\right)\frac{\partial \Delta x_1}{\partial x_i}dA$$

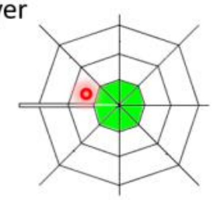


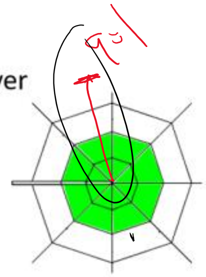
289

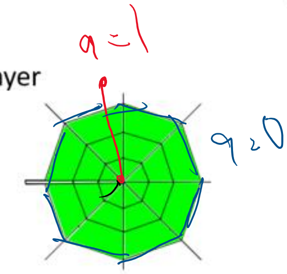
J integral: 2. EDI FEM Aspects

- Since $J_0 \to 0$ the inner J_0 collapses to the crack tip (CT)
- J_1 will be formed by element edges
- By using spider web (rozet) meshes any <u>reasonable</u> number of layers can be used to compute J:









General form of J

$$J=\int_{A^*}\left\{\left[\sigma_{ij}\frac{\partial u_j}{\partial x_1}-w\delta_{1i}\right]\frac{\partial q}{\partial x_i}+\left[\sigma_{ij}\frac{\partial \varepsilon_{ij}^p}{\partial x_1}-\frac{\partial w^p}{\partial x_1}+\alpha\sigma_{ii}\frac{\partial \Theta}{\partial x_1}-F_i\frac{\partial u_j}{\partial x_1}\right]q\right\}dA-\int_{\Gamma_++\Gamma_-}\sigma_{2j}\frac{\partial u_j}{\partial x_1}q\,d\Gamma$$

Plasticity effects

Thermal effects

Body force

Nonzero crack surface traction

Simplified Case:

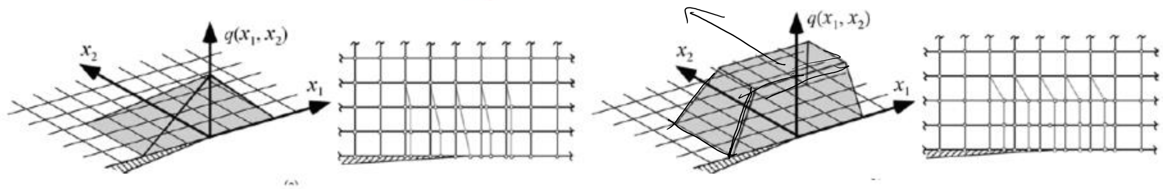
(Nonlinear) elastic, no thermal strain, no body force, traction free crack surfaces

$$\boxed{J=\int_{A^*}\left[\sigma_{ij}\frac{\partial u_j}{\partial x_1}-w\delta_{1i}\right]\frac{\partial q}{\partial x_i}dA}$$

J integral: 2. EDI FEM Aspects

- Shape of decreasing function q:

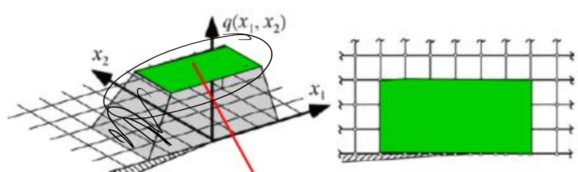
q≈1



Pyramid q function

Plateau q function

- Plateau q function useful when inner elements are not very accurate: e.g. when singular/quarter point elements are not used



$$\frac{\partial q}{\partial x_i} = 0$$ These elements do not contribute to J

$$J = \int_{A^*} \left[\sigma_{ij} \frac{\partial u_j}{\partial x_1} - w\delta_{1i} \right] \frac{\partial q}{\partial x_i} dA$$

6.1.5. Finite Element mesh design for fracture mechanics

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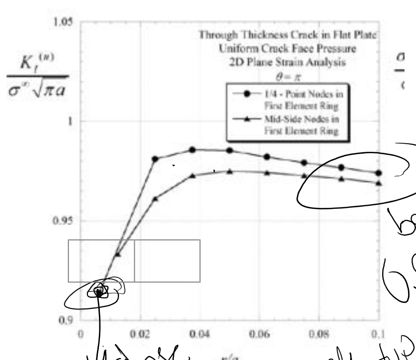
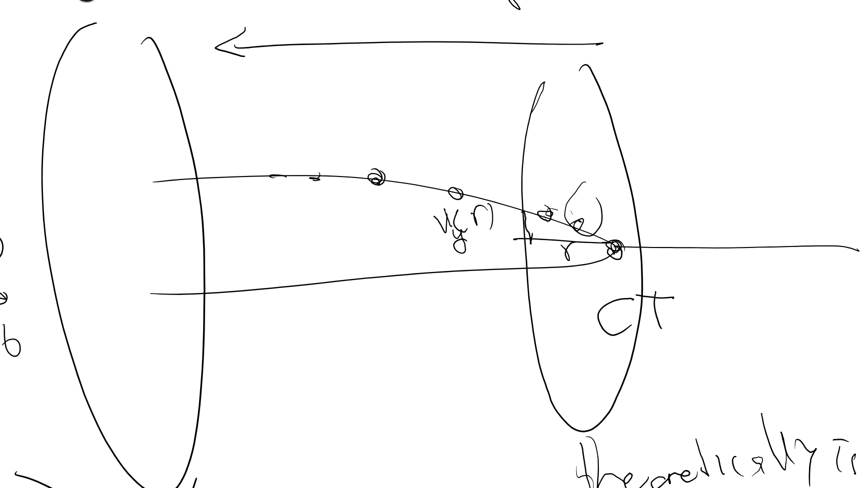


bad 0.96

close to the crack tip

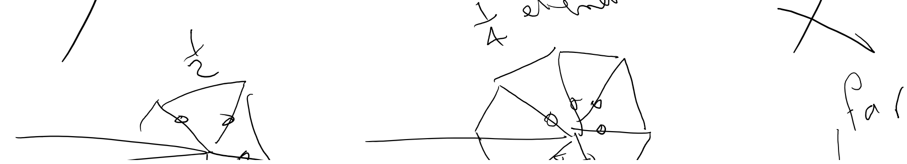


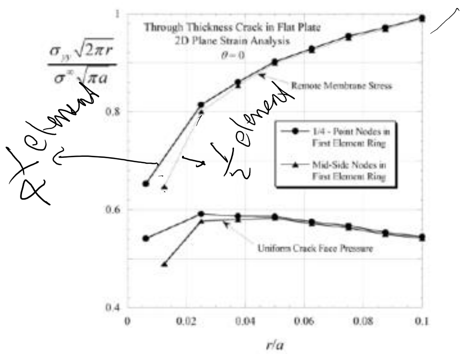
K from displacement u

$$K_I = \lim_{r \to 0} \left[\frac{E' u_y}{4} \sqrt{\frac{2\pi}{r}} \right] \quad (\theta = \pi)$$

theoretically is good

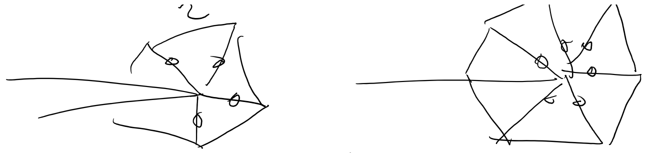
numerically

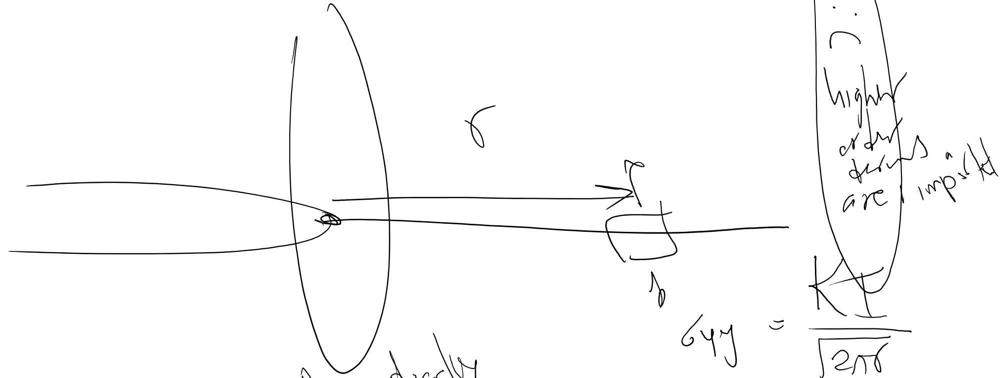




K from stress σ

$$K_I = \lim_{r \longrightarrow 0} \left(\sqrt{2\pi r}\ \sigma_{22}|_{\theta=0} \right)$$





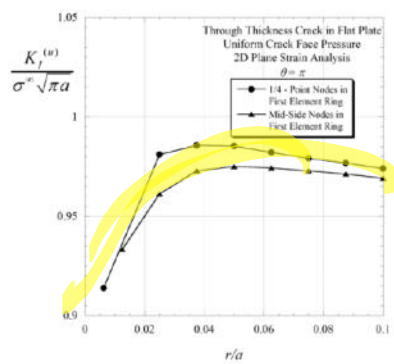
$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$

$K_I = \sigma_{yy}\sqrt{2\pi r}$

mathematically :)

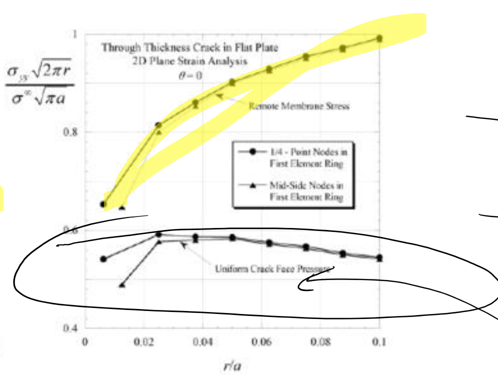
numerically :(

disp is better ✓



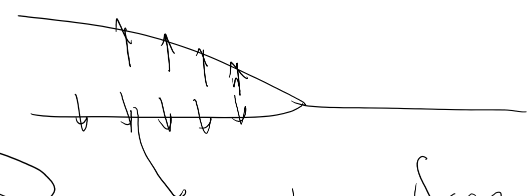
K from displacement ***u***

$$K_I = \lim_{r \to 0} \left[\frac{E' u_y}{4} \sqrt{\frac{2\pi}{r}} \right] \quad (\theta = \pi)$$



K from stress σ

$$K_I = \lim_{r \longrightarrow 0} \left(\sqrt{2\pi r}\ \sigma_{22}|_{\theta=0} \right)$$



crack surface

but $u_j \neq 0$