2024/11/25

Monday, November 25, 2024 9:42 AM

4.3 Mixed mode fracture

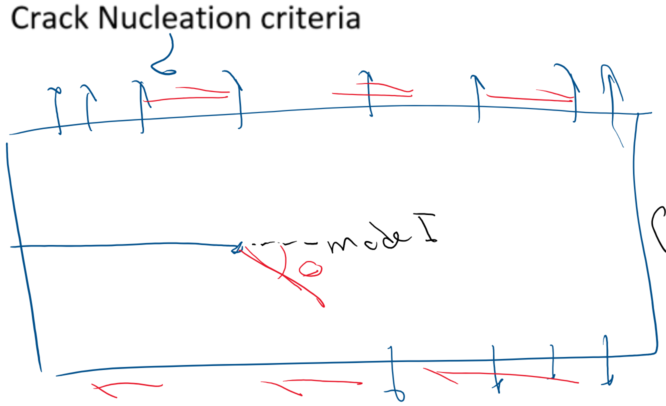
4.3.1 Crack propagation criteria

a) Maximum Circumferential Tensile Stress

b) Maximum Energy Release Rate

c) Minimum Strain Energy Density

4.3.2 Crack Nucleation criteria

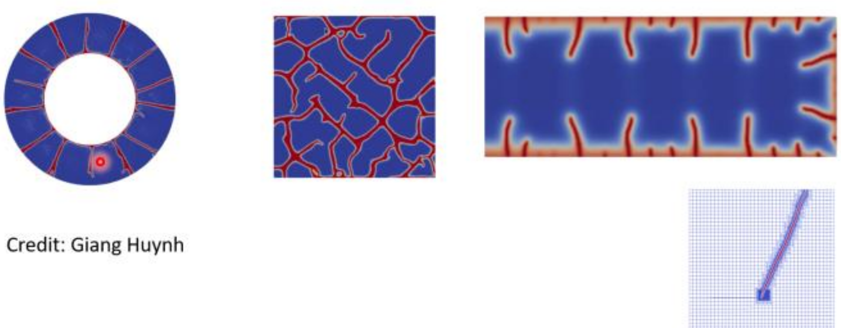


quasi-static

$G \geq R$

crack growth

For bulk models we don't need to answer crack propagation direction directly



Credit: Giang Huynh

Bulk models do not need a propagation direction criterion (and to some extent nucleation)

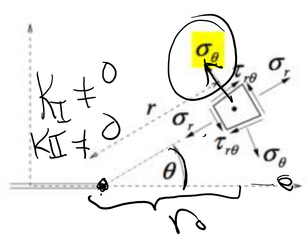
Sharp crack models need such criteria:

LEFM, TSR

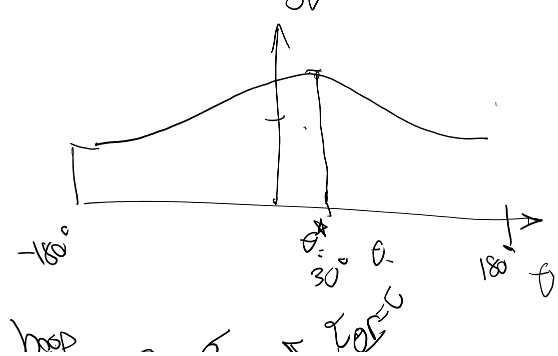
today

Maximum circumferential stress criterion

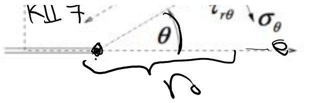
Erdogan and Sih



maximum circumferential stress criterion (maximum hoop stress criterion):
crack propagates in the direction perpendicular to the maximum circumferential stress
(evaluated on a circle of a small diameter centered at the tip)



hoop $\tau_{r\theta} = 0$



the direction of propagation
is given by the angle θ_c for which

maximum circumferential stress
(evaluated on a circle of a small diameter centered at the tip)

$$\sigma_\theta(r,\theta_c) = \max_{-\pi<\theta<\pi} \sigma_\theta(r,\theta)$$

(from M. Jirasek)

principal stress → $\boxed{\tau_{r\theta} = 0}$

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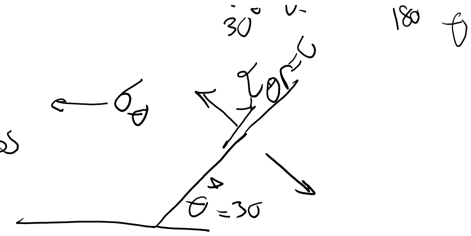
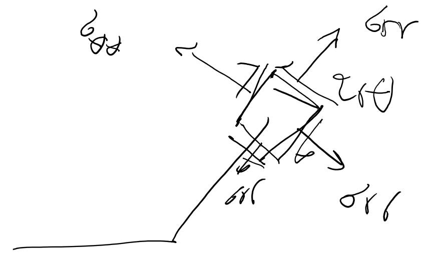
hoop stress



Maximum circumferential stress criterion



$$\sigma_r = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right) \tag{7.35a}$$

$$\sigma_\theta = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right) \tag{7.35b}$$

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right). \tag{7.35c}$$

$$\tau_{r\theta} = 0 \longrightarrow \quad K_I\left(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right) + K_{II}\left(\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right) = 0$$

$$\longrightarrow \quad \boxed{\theta_c = 2\arctan\frac{1}{4}\left(K_I/K_{II} \pm \sqrt{(K_I/K_{II})^2 + 8}\right)}$$

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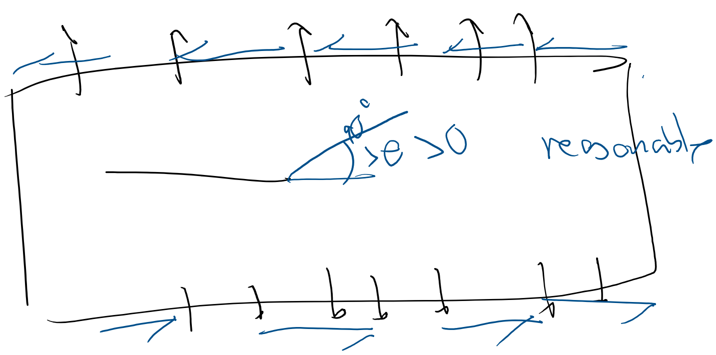
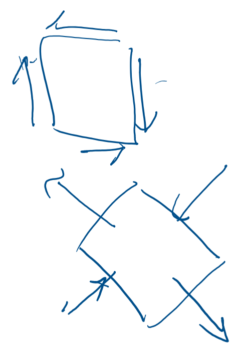
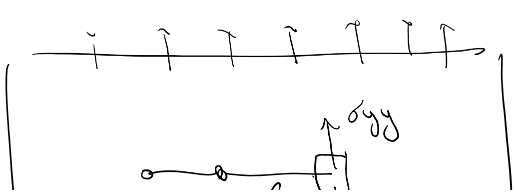
angle of crack propagation



reasonable

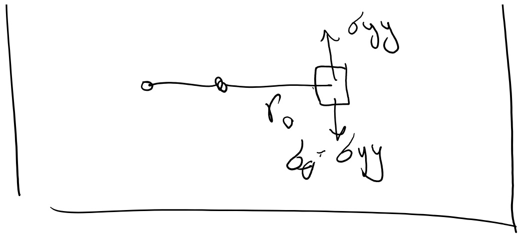


crack growth



$G = R$

K_I^2 D



$G = R$

$$\frac{K_I^2}{E'} = R \Rightarrow K_I = \sqrt{RE'}$$

$$\sigma_{yy}(r_0) = \frac{K_I}{\sqrt{\pi r_0}} = \frac{\sqrt{RE'}}{\sqrt{\pi r_0}}$$

so at distance r_0 $\quad \sigma_{\theta\theta} \geq \frac{\sqrt{RE'}}{\sqrt{\pi r_0}}$ corresponds to crack growth

Maximum allowable traction $\sigma_{\theta max}$ is reached at angle $\theta = \theta max$ and distance from crack tip r_0:

$$\sigma_{\theta max}\sqrt{2\pi r_0} = K_{\mathrm{Ic}} = \cos\frac{\theta_0}{2}\left[K_{\mathrm{I}}\cos^2\frac{\theta_0}{2} - \frac{3}{2}K_{\mathrm{II}}\sin\theta_0\right]$$

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Maximum circumferential stress criterion

Fracture criterion $\quad K_{eq} \geq K_{Ic}$

$$\sigma_\theta = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}}\cos\frac{\theta_0}{2}\left(1 - \sin^2\frac{\theta_0}{2}\right) + \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta_0}{2} - \frac{3}{4}\sin\frac{3\theta_0}{2}\right) \tag{7.9}$$

must reach a critical value which is obtained by rearranging the previous equation

$$\sigma_{\theta max}\sqrt{2\pi r} = K_{\mathrm{Ic}} = \cos\frac{\theta_0}{2}\left[K_{\mathrm{I}}\cos^2\frac{\theta_0}{2} - \frac{3}{2}K_{\mathrm{II}}\sin\theta_0\right] \tag{7.10}$$

which can be normalized as

Ex

Find theta for crack propagation:

$$\boxed{\frac{K_{\mathrm{I}}}{K_{\mathrm{Ic}}}\cos^3\frac{\theta_0}{2} - \frac{3}{2}\frac{K_{\mathrm{II}}}{K_{\mathrm{Ic}}}\cos\frac{\theta_0}{2}\sin\theta_0 \geq 1} \tag{7.11}$$

11 This equation can be used to define an equivalent stress intensity factor K_{eq} for mixed mode problems

$$\boxed{K_{eq} = K_{\mathrm{I}}\cos^3\frac{\theta_0}{2} - \frac{3}{2}K_{\mathrm{II}}\cos\frac{\theta_0}{2}\sin\theta_0} \tag{7.12}$$

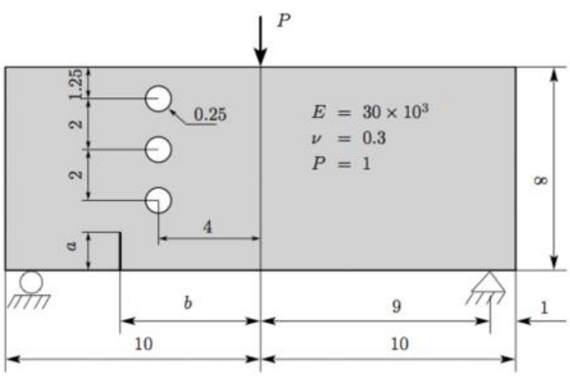
$K_{eq} \geq K_{Ic}$ for crack growth

Step 1 : Find potential crack propagation angle:

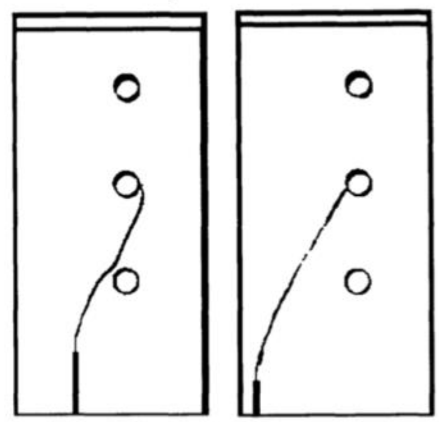
$$\theta_0 = 2 \arctan \frac{1}{4}\left(K_I/K_{II} \pm \sqrt{(K_I/K_II)^2+8}\right)$$

Step 2: Calculate equivalent K (Keq) and see if it's larger than KIc:

$$\boxed{K_{eq} = K_{\mathrm{I}} \cos^3 \frac{\theta_0}{2} - \frac{3}{2} K_{\mathrm{II}} \cos \frac{\theta_0}{2} \sin \theta_0} \geq K_{Ic} \qquad (7.12)$$ it grows



Experiment



XFEM

$$\theta_c = 2 \arctan \frac{1}{4}\left(K_I/K_{II} \pm \sqrt{(K_I/K_II)^2+8}\right)$$

4.3 Mixed mode fracture

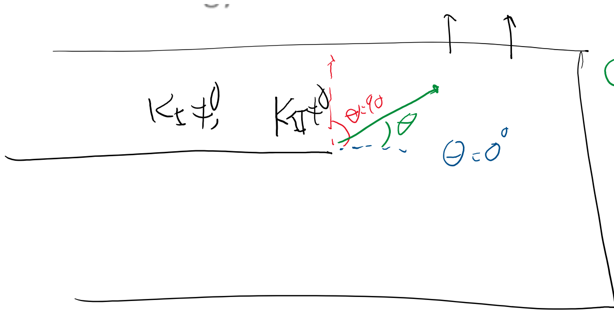
4.3.1 Crack propagation criteria

a) Maximum Circumferential Tensile Stress

b) Maximum Energy Release Rate

$G(\theta=90) = J_2 = 2\frac{K_I K_{II}}{E'}$

$G(\theta) = \ldots$



$G(\theta) = \ldots$

$J_1 = G = \frac{K_I^2 + K_{II}^2}{E'}$ $\theta = 0°$

Most dissipation possible:

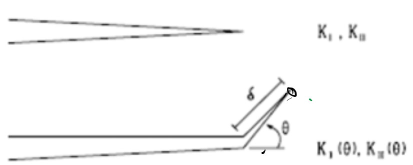
Maximum Energy Release Rate (MERR)

G: crack driving force -> crack will grow in the direction that G is maximum

Erdogan, F. and Sih, G.C. 1963

"If we accept Griffith (energy) theory as the valid criteria which explains crack growth, then the crack will grow in the direction along which the elastic energy release per unit crack extension will be maximum and the crack will start to grow when this energy reaches a critical value (or $G = G(\delta, \theta)$). Evaluation of $G(\delta, \theta)$ poses insurmountable mathematical difficulties."

Maximum Energy Release Rate



Stress intensity factors for **kinked crack extension**:
Hussain, Pu and Underwood (Hussain et al. 1974)

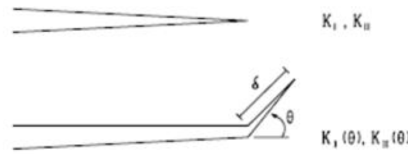
$$\begin{Bmatrix} K_I(\theta) \\ K_{II}(\theta) \end{Bmatrix} = \left(\frac{4}{3+\cos^2\theta}\right)\left(\frac{1-\frac{\theta}{\pi}}{1+\frac{\theta}{\pi}}\right)^{\frac{\theta}{2\pi}} \begin{Bmatrix} K_I\cos\theta + \frac{3}{2}K_{II}\sin\theta \\ K_{II}\cos\theta - \frac{1}{2}K_I\sin\theta \end{Bmatrix}$$

$$G(\theta) = \frac{1}{E'}\left(K_I^2(\theta) + K_{II}^2(\theta)\right)$$

$$G(\theta) = \frac{4}{E'}\left(\frac{1}{3+\cos^2\theta}\right)^2\left(\frac{1-\frac{\theta}{\pi}}{1+\frac{\theta}{\pi}}\right)^{\frac{\theta}{\pi}} \left[(1+3\cos^2\theta)K_I^2 + 8\sin\theta\cos\theta K_I K_{II} + (9-5\cos^2\theta)K_{II}^2\right]$$

Maximization condition

$$\frac{\partial G(\theta)}{\partial \theta} = 0$$
$$\frac{\partial^2 G(\theta)}{\partial \theta^2} < 0$$



$$G(\theta) = \frac{4}{E'}\left(\frac{1}{3+\cos^2\theta}\right)^2\left(\frac{1-\frac{\theta}{\pi}}{1+\frac{\theta}{\pi}}\right)^{\frac{\theta}{\pi}}$$
$$[(1+3\cos^2\theta)K_I^2 + 8\sin\theta\cos\theta K_I K_{II} + (9-5\cos^2\theta)K_{II}^2]$$

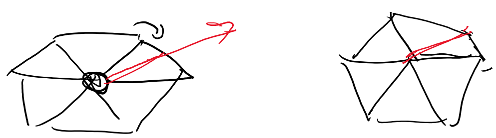
we find θ_0 that maximizes G from this eqn

$$\boxed{4\left(\frac{1}{3+\cos^2\theta_0}\right)^2\left(\frac{1-\frac{\theta_0}{\pi}}{1+\frac{\theta_0}{\pi}}\right)^{\frac{\theta_0}{\pi}} \left[\left(1+3\cos^2\theta_0\right)\left(\frac{K_{\mathrm{I}}}{K_{\mathrm{Ic}}}\right)^2 + 8\sin\theta_0\cos\theta_0\left(\frac{K_{\mathrm{I}}K_{\mathrm{II}}}{K_{\mathrm{Ic}}^2}\right) + \left(9-5\cos^2\theta_0\right)\left(\frac{K_{\mathrm{II}}}{K_{\mathrm{Ic}}}\right)^2\right] = 1}$$

$K_I = 5 K_{II}$

$\rightarrow \quad \theta_0 = 25°$ (hypothetical)

$$G(\theta_0) = \underset{\text{exp}}{\frac{4}{E'}}\left(\frac{1}{3+\cos^2\theta_0}\right)^2 \cdots \underbrace{\geq R}_{\text{crack grows}}$$



Strain Energy Density (SED) criterion

Sih 1973

$$U_i = \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij} \quad U_i = \frac{1}{4\mu}\left[\frac{\kappa+1}{4}(\sigma_x^2+\sigma_y^2) - 2(\sigma_x\sigma_y - \tau_{xy}^2)\right]$$

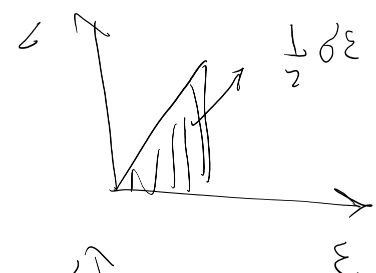
$$\sigma_x = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) - \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$\sigma_y = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \quad (7.13)$$

$$\tau_{xy} = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} + \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right).$$

$$S = rU_i$$

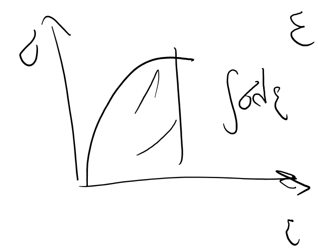
$\boxed{\frac{1}{2}\sigma:\varepsilon \quad \text{2D : 3D linear}}$ LEFM



$\sigma \propto \frac{1}{\sqrt{r}}$

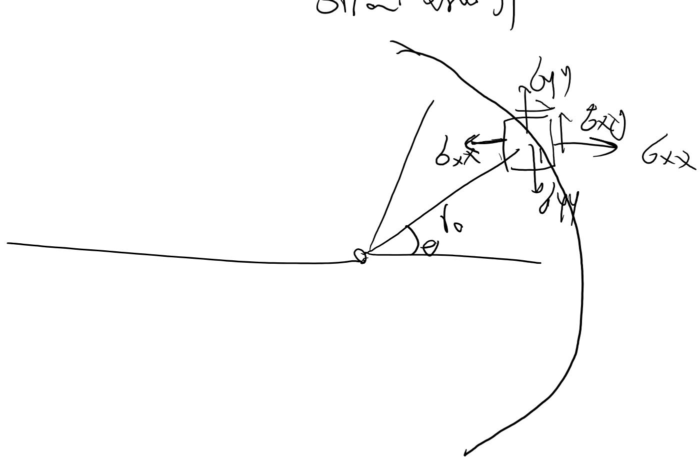
$\int_0^{\varepsilon_f} \sigma : d\varepsilon$

$\varepsilon \propto \frac{1}{\sqrt{r}}$



$U \propto \frac{S}{r}$

strain energy



$$U_i(r_0,\theta) = \frac{S(\theta)}{r_0}$$

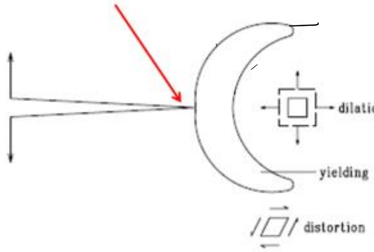
Strain Energy Density (SED) criterion

- Crack direction θ_0 which **minimizes** the strain energy density S
- Crack Extends when S reaches a critical value at a distance r_0

Minimization condition

$$\frac{\partial S}{\partial \theta} = 0$$
$$\frac{\partial^2 S}{\partial \theta^2} > 0$$

Pure mode I (0 degree has smallest S)



Find theta that minimizes S (I have not provided the formula for this)
Then, check if the crack actually propagates:

Strain Energy Density (SED) criterion

θ found

$$\boxed{\frac{8\mu}{(\kappa-1)}\left[a_{11}\left(\frac{K_{\mathrm{I}}}{K_{\mathrm{Ic}}}\right)^2 + 2a_{12}\left(\frac{K_{\mathrm{I}}K_{\mathrm{II}}}{K_{\mathrm{Ic}}^2}\right) + a_{22}\left(\frac{K_{\mathrm{II}}}{K_{\mathrm{Ic}}}\right)^2\right] \geq 1}$$

crack propagates

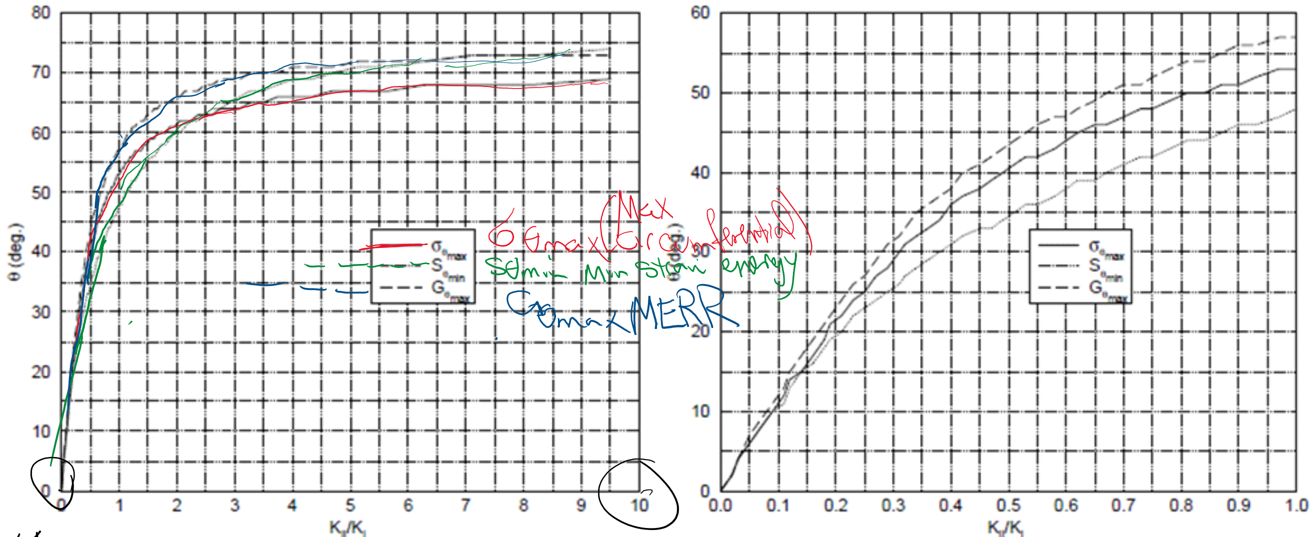
$$a_{11} = \frac{1}{16\mu}\left[(1+\cos\theta)(\kappa-\cos\theta)\right]$$
$$a_{12} = \frac{\sin\theta}{16\mu}\left[2\cos\theta-(\kappa-1)\right]$$
$$a_{22} = \frac{1}{16\mu}\left[(\kappa+1)(1-\cos\theta)+(1+\cos\theta)(3\cos\theta-1)\right]$$

$$\kappa = \frac{3-\nu}{1+\nu} \quad \text{(plane stress)}$$
$$\kappa = 3-4\nu \quad \text{(plane strain)}$$

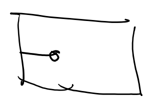
We can refer to graphs to decide:

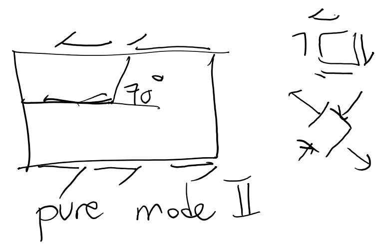
1) In what direction does the crack grow?
2) What is the direction of crack growth?

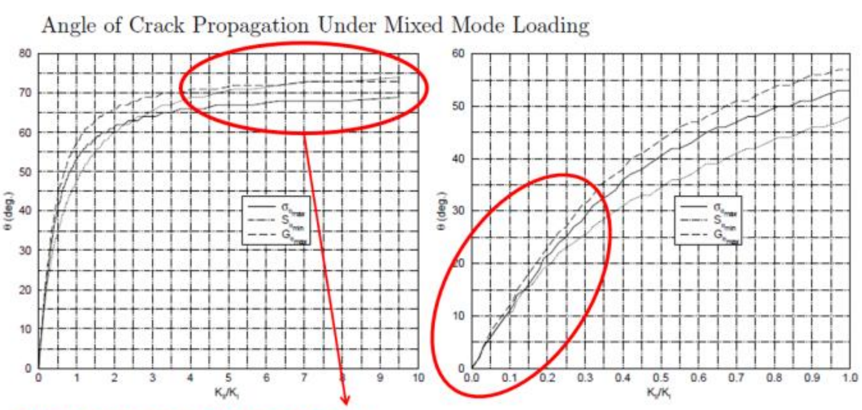
① propagat.



$\frac{K_{II}}{K_I}$, 0 $\quad$ K_{II}/K_I $\quad$ 1.0







~ 70 degree angle for mode II ! $\quad$ Zoom view (low K_{II} component)

Good agreement for low K_{II}

Does the crack grow?

Comparison:

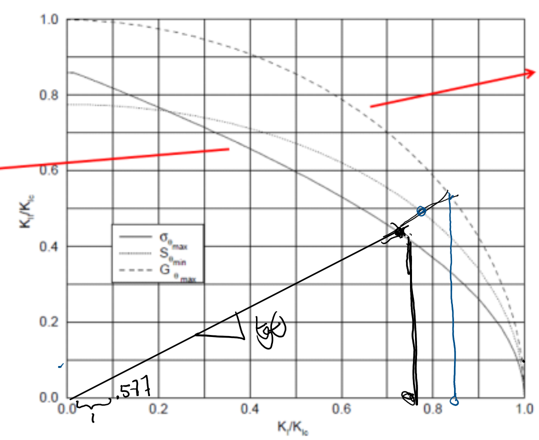
b) Locus of crack propagation

Does the crack grow?

Comparison:
b) Locus of crack propagation

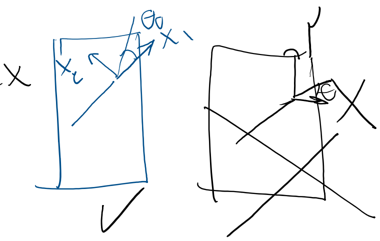
Most conservative

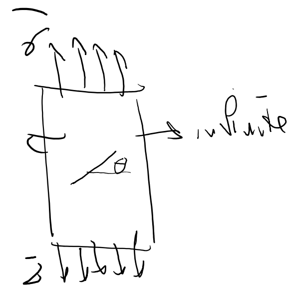
$\sigma_{\theta max}$



Least conservative

$G_{\theta max}$





Example: $K_I = \sqrt{\pi a}\,\bar{\sigma}\cos^2\theta$, $K_{II} = \sqrt{\pi a}\,\bar{\sigma}\sin\theta\cos\theta$ $\Rightarrow$ $\boxed{\dfrac{K_{II}}{K_I} = tg\,\theta}$

$\theta = 30^\circ$ $\quad tg\,\theta = \dfrac{1}{\sqrt{3}} = .577$

for $\sigma_{\theta max}$: $\dfrac{K_I}{K_{Ic}} = .75 \Rightarrow \dfrac{\sqrt{\pi a}\,\bar{\sigma}\cos^2\theta}{K_{Ic}} = .75 \Rightarrow (\bar{\sigma}_{max})_{\sigma_{\theta max}} = \dfrac{K_{Ic}}{\sqrt{a}}\left(\dfrac{.75}{\sqrt{\pi}\cos^2\theta}\right)$

$3/4$

for $G_{\theta max}$: $\dfrac{K_I}{K_{Ic}} = .835 \longrightarrow (\bar{\sigma}_{max})_{G_{\theta max}} = \dfrac{K_{Ic}}{\sqrt{a}}\left(\dfrac{.835}{\sqrt{\pi}\,\frac{3}{4}}\right)$

$$(\bar{\sigma}_{max})_{\sigma_{\theta max}} < (\bar{\sigma}_{max})_{G_{\theta max}}$$

FYI Extention to cohesive models, ..., non-LEFM

$\sigma_{eff}(\sigma_n, \tau)$

θ_0 corresponds to where σ_{eff} is max one model $\sigma_{eff} = \sqrt{(\langle \sigma_n \rangle_+)^2 + (\alpha \tau)^2}$

circumferential stress criterion

- **Effective traction** can be defined as a function of both normal σ_θ and tangential $\tau r\theta$ components of traction. For example:

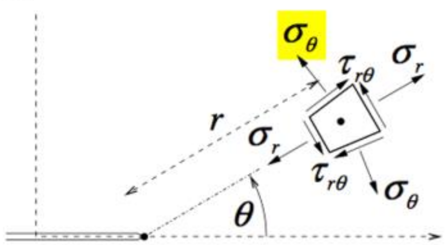
$$\sigma_{\text{eff}} = \sqrt{\sigma_\theta^2 + (\alpha\tau_{r\theta})^2}$$

combines normal and tangential components through **mode mixity parameter** α.

- Crack propagation direction θ_c can be based on maximizing effective traction:

$$\sigma_{\text{eff}}(r, \theta_c) = \max_{-\pi<\theta<\pi} \sigma_{\text{eff}}(r, \theta)$$

- For example, in soil and rock applications normal tractions can be compressive for cracks that propagate under high shear tractions.



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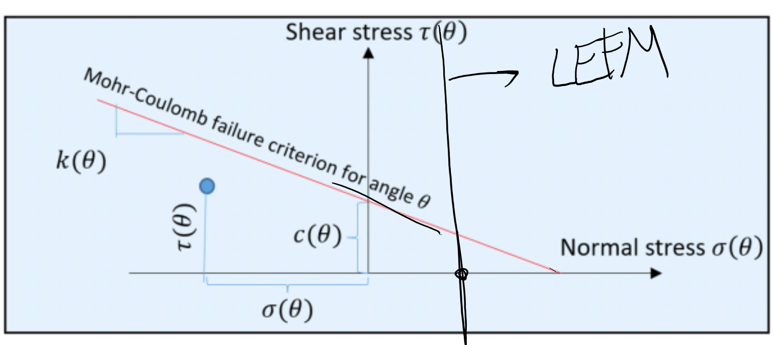
Modifications to maximum circumferentia stress criterion: MC criterion

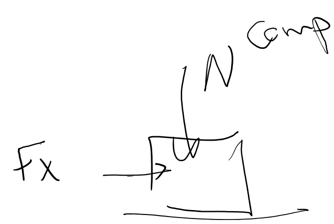


Checking failure along plane with angle θ:

$\sigma_{eff} = \tau(\theta) + k(\theta)\sigma(\theta) \geq c(\theta)$

Angle-dependent stress > angle-dependent strength

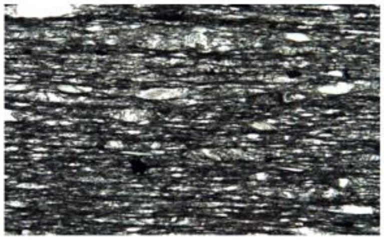
$\sqrt{(\alpha\tau)^2 \; (\langle\sigma_n\rangle_+)^2}$



friction coefficient K

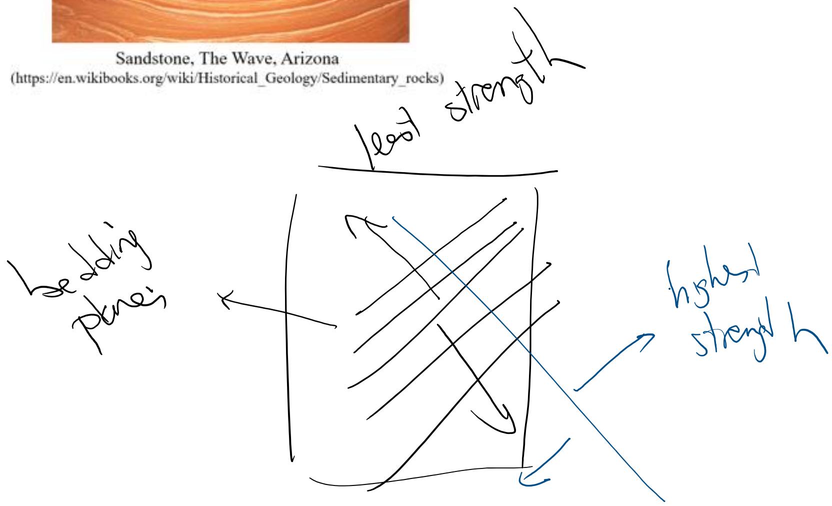
$F_x \geq NK + c$

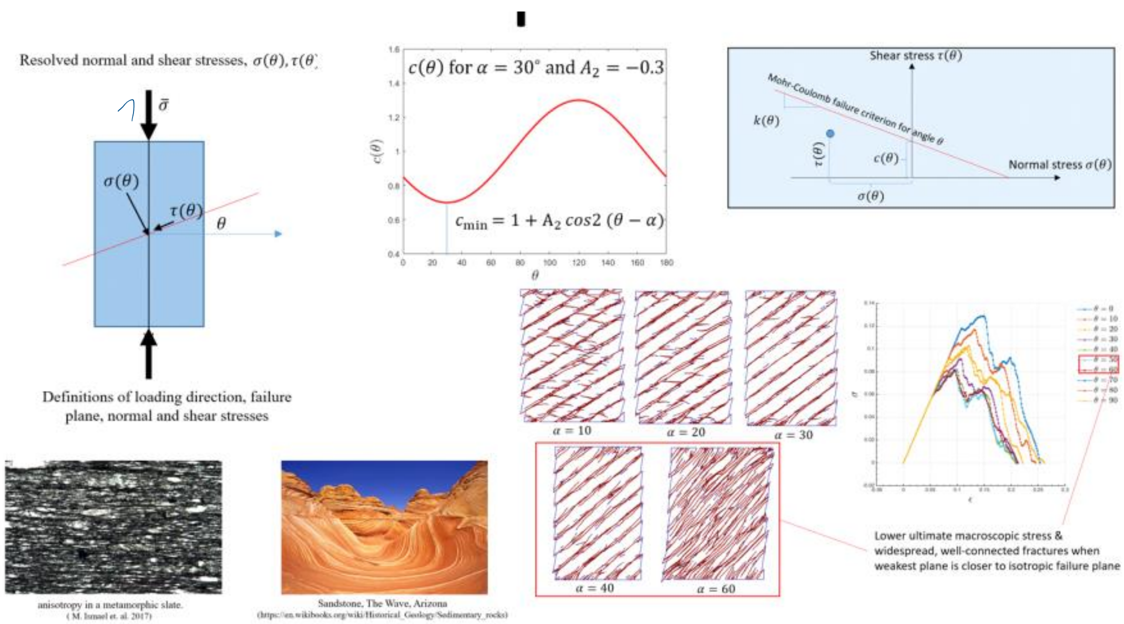
cohesion



anisotropy in a metamorphic slate.
(M. Ismael et. al. 2017)

Sandstone, The Wave, Arizona
(https://en.wikibooks.org/wiki/Historical_Geology/Sedimentary_rocks)





In the absence of a major crack we need to have a nucleation criterion too

Crack nucleation criterion

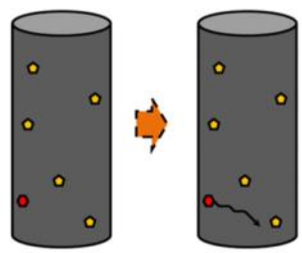
- Cracks nucleate from microscopic material defects under high stress/ strain loads.
- For each crack propagation criterion there can be a corresponding nucleation criterion.
- For example for maximum circumferential tensile stress, a crack nucleates when the maximum principle stress σ_1 at a point reaches material strength σ_0:

$$\max_{-\pi<\theta<\pi}\sigma_\theta(r \to 0^+, \theta) = \sigma_1 = \sigma_0, \quad \text{crack nucleates}$$

Although we assume that there is no initial crack tip, we can measure r relative to the potential nucleation point.

- Same concept applies to modified maximum circumferential tensile stress criteria:

$$\max_{-\pi<\theta<\pi}\sigma_{\text{eff}}(r \to 0^+, \theta) = \sigma_0, \quad \text{crack nucleates}$$



Crack nucleation criterion

- For Maximum Energy Release Rate Criterion if we assume there are no defects, there will be no crack nucleation. However, assuming that local stress field generates a tensile maximum principal stress of σ_1 a "microscopic" initial crack (defect) of length a_{ini} perpendicular to σ_1 direction generates,

$$G = \frac{K_I^2 + K_{II}^2}{E'} = \pi a_{\text{ini}}\sigma_1^2$$

so the microcrack propagates (*i.e.*, a "macroscopic" crack nucleates) when,

$$G = G_c \quad \Leftrightarrow \quad \sigma_1 = \sqrt{\frac{G_c}{\pi a_{\text{ini}}}}$$

  - Initial crack direction perpendicular to σ_1 is chosen to maximize G.
  - We have assumed the initial crack to be small enough to use the infinite domain SIF formula of $K_I = \sqrt{\pi a}\sigma$.